Lecture 6 - S-Parameters

Microwave Active Circuit Analysis and Design

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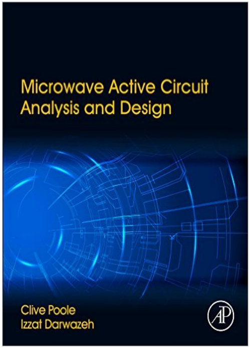

Learning Objectives

- Knowledge
  - Be aware of the limitations of immittance parameters at higher frequencies.
  - Understand the definition of S-parameters as ratios of 'power waves' at the ports of a network.
  - Understand the principles of S-parameter measurement.
  - Understand the need for calibration to remove error.
  - Understand the various multi-term error models and how they are applied.
- Skills
  - Be able to calculate the input reflection coefficient of a two-port network with an arbitrary load termination, and the output reflection coefficient of a two-port network with an arbitrary source termination.
  - Be able to apply S-parameters to determine whether a given two-port is 'active', 'passive' or 'lossless'.
  - Be able to apply Mason's figure of merit to determine the degree of *Unilaterality* of a network.
  - Be able to calculate the S-parameters of two cascaded two-port networks by applying the T-parameters.
  - Be able to apply the Signal Flow Graph technique as a useful means of visualising and solving S-parameters network problems.
  - Be able to, with suitable training, carry out a set of S-parameter measurements using a vector network analyser.

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Definition of the scattering parameters

With reference to figure 1 the instantaneous port terminal voltage and current are related to the incident and reflected voltage and current components by the following relationships:

$$V_1 = V_{inc} + V_{ref} \qquad (1)$$

$$I_1 = I_{inc} - I_{ref} \qquad (2)$$

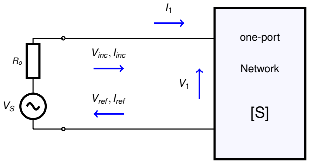


Figure 1 : One-port network with power waves

If one considers the case of a transmission line system having a real characteristic impedance, R_o, then the incident and reflected components are related to this characteristic impedance by:

$$\frac{V_{inc}}{I_{inc}} = \frac{V_{ref}}{I_{ref}} = R_o \qquad (3)$$

Definition of the scattering parameters

Combining equations (1), (2) and (3) yields the following relationships:

$$\left.\begin{aligned} V_{inc} &= \frac{1}{2}(V_1 + R_o I_1) \\ V_{ref} &= \frac{1}{2}(V_1 - R_o I_1) \end{aligned}\right\} \quad (4)$$

$$\left.\begin{aligned} I_{inc} &= \frac{(V_1 + R_o I_1)}{2R_o} \\ I_{ref} &= \frac{(V_1 - R_o I_1)}{2R_o} \end{aligned}\right\} \quad (5)$$

Either of the pairs of equations (4) or (5) are sufficient to fully describe the one-port of figure 1, but these may be replaced by a single pair of equations if we introduce the concept of normalised current and voltage variables[5]. From equations (4) and (5), let us define two new normalised variables *a* and *b* such that:

$$\frac{V_{inc}}{\sqrt{R_o}} = I_{inc}\sqrt{R_o} = a \quad (6)$$

$$\frac{V_{ref}}{\sqrt{R_o}} = I_{ref}\sqrt{R_o} = b \quad (7)$$

Definition of the scattering parameters

The variables a and b have the dimensions of the square root of power and are known as the *scattering variables*. It is more useful to define these variables in terms of the terminal voltages and currents, V_1 and I_1, so by combining equations (4) to (7) we can now write:

$$a = \frac{V_1 + I_1 Z_o}{2\sqrt{R_o}} \tag{8}$$

$$b = \frac{V_1 - I_1 Z_o}{2\sqrt{R_o}} \tag{9}$$

The discussion so far has been based on the case of a real reference impedance, R_o. Penfield[12] extended the definition of the variables a and b by considering the case of a complex reference impedance, Z_o. The variables were thus renamed *power waves* and redefined as follows:

$$a = \frac{V + IR_o}{2\sqrt{|Re(Z_o)|}} \tag{10}$$

$$b = \frac{V - IR_o}{2\sqrt{|Re(Z_o)|}} \tag{11}$$

Let us define a dimensionless ratio 'S' for the one-port of figure 1 as:

$$S = \frac{a}{b} \tag{12}$$

Definition of the scattering parameters

The one-port relationship in equation (12) can be extended to the two-port network shown in figure 2 by replacing *a* and *b* by the column vectors [*a*] and [*b*]. These power wave vectors are related to each other by an *S*-matrix containing four complex parameters, as follows:

$$\left.\begin{aligned} b_1 &= S_{11}a_1 + S_{12}a_2 \\ b_2 &= S_{21}a_1 + S_{22}a_2 \end{aligned}\right\} \tag{13}$$

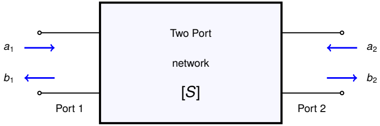


Figure 2 : Two-port network with power waves

Or, using matrix notation :

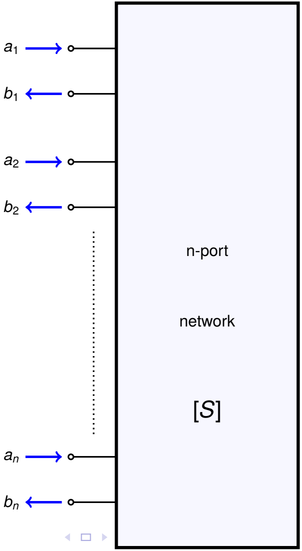
$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \tag{14}$$

Definition of n-port scattering parameters

The S-matrix representation can be extended to describe a network with n pairs of terminals by considering the incident and reflected power waves at each port, as shown in figure 3.

In the general case, for any network with n-ports, the power wave variables defined by (10) and (11) are expressed as n^{th} order column vectors, $[a]$ and $[b]$, which are related by the $n \times n$ S-matrix as follows.-

$$\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & \cdots & S_{1n} \\ S_{21} & S_{22} & \cdots & S_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ S_{n1} & S_{n2} & \cdots & S_{nn} \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \tag{15}$$



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Input and output impedance of a two-port

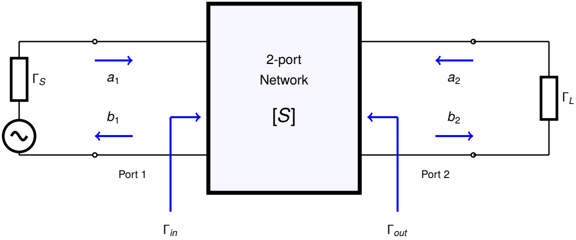

Figure 4 : Two-port network with arbitrary source and load

In the general case of a two-port network where $S_{12} \neq 0$ and $S_{21} \neq 0$, the input reflection coefficient Γ_{in} will be dependent on the value of the load and vice versa. To derive the input reflection coefficient of the two-port, Γ_{in}, with load Γ_L terminating port 2, we start by stating the following power wave relationship at the load :

$$a_2 = b_2\Gamma_L \tag{16}$$

Input and output impedance of a two-port

If we now substitute (16) into (13) we obtain the following :

$$b_2 = S_{21}a_1 + S_{22}b_2\Gamma_L \tag{17}$$

Which can be rearranged to isolate b_2 as follows :

$$b_2 = \frac{S_{21}a_1}{(1 - S_{22}\Gamma_L)} \tag{18}$$

The input reflection coefficient of the two-port is defined as $\Gamma_{in} = b_1/a_1$, or, by applying (13) and (16) :

$$\Gamma_{in} = S_{11} + \frac{S_{12}b_2\Gamma_L}{a_1} \tag{19}$$

Simply combining (18) and (19) yields the following :

$$\boxed{\Gamma_{in} = S_{11} + \frac{S_{12}S_{21}\Gamma_L}{1 - S_{22}\Gamma_L}} \tag{20}$$

Input and output impedance of a two-port

Given the definition of $\Delta = S_{11}S_{22} - S_{12}S_{12}$ the reader can easily show that (20) can also be written as:

$$\Gamma_{in} = \frac{S_{11} - \Delta\Gamma_L}{1 - S_{22}\Gamma_L} \tag{21}$$

Both forms of this expression (i.e. (20) and (21)) are found in general use.
A similar analysis applied to the output port yields the expression for the output reflection coefficient with an arbitrary source termination :

$$\boxed{\Gamma_{out} = S_{22} + \frac{S_{12}S_{21}\Gamma_S}{1 - S_{11}\Gamma_S}} \tag{22}$$

Which can likewise also be written as :

$$\Gamma_{out} = \frac{S_{22} - \Delta\Gamma_S}{1 - S_{11}\Gamma_S} \tag{23}$$

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Losseless and reciprocal networks

Let P_{in} and P_{out} represent total input and output powers, respectively. These powers are defined, for an n port network, as:

$$P_{in} = \sum_{i=1}^{n} |a_i|^2 = [a]^\dagger [a] \tag{24}$$

$$P_{out} = \sum_{i=1}^{n} |b_i|^2 = [b]^\dagger [b] \tag{25}$$

Where $[M]^\dagger$ represents the conjugate transpose (Hermitian matrix) of a given matrix M. Given that for a lossless network $P_{in} = P_{out}$ by definition, we combine (24) and (25) as follows:

$$[a]^\dagger [a] = [b]^\dagger [b] \tag{26}$$

Noting that $b = [S]a$ we can rewrite (26) as:

$$[a]^\dagger [I][a] = ([S][a])^\dagger ([S][a]) \tag{27}$$

Where $[I]$ is the identity matrix. (27) can be rearranged as:

$$[a]^\dagger [I][a] = [a]^\dagger [S]^\dagger [S][a] \tag{28}$$

Losseless and reciprocal networks

Which implies that:

$$[S]^{\dagger}[S] = [I] \tag{29}$$

A consequence of (29) for an $n \times m$ S-matrix is that:

$$\sum_{i=1}^{n} |S_{im}|^2 = 1 \tag{30}$$

for all m

A reciprocal network is a multi-port network in which the power losses are the same between any pair of ports regardless of direction of propagation. In S-parameter terms, this means $S_{ij} = S_{ji}$ for any $i \neq j$. Most passive networks such as cables, attenuators, power dividers and couplers are reciprocal.

Activity and Passivity

the squared magnitude of the incident power wave, a, is equal to the available power from a linear generator connected to the i^{th} port of the network, i.e.:

$$P_n = |a_i|^2 \tag{31}$$

Similarly, the reflected power from the i^{th} port can be written as:

$$P_r = |b_i|^2 \tag{32}$$

Thus in terms of power wave variables, the power delivered to a one-port network is given by : -

$$P = |a_i|^2 - |b_i|^2 \tag{33}$$

The ratio of reflected to incident power at the device port is known as the power reflection coefficient which is equal to the squared magnitude of the S-parameters

$$\frac{|b_i|^2}{|a_i|^2} = |S|^2 \tag{34}$$

Activity and Passivity

Using the above relationship, equation (33) for the power flowing into the one-port network can be written in terms of the one-port scattering parameter as follows:

$$P = |a|^2 \left(1 - |S|^2\right) \tag{35}$$

The relationships discussed so far can be generalised to deal with the case of the n-port network by considering the power flow at each port. The power flowing into the i^{th} port is described by an equation of the same form as equation (35) for that particular port:

$$P = |a_i|^2 - |b_i|^2 = a_i a_i^* - b_i b_i^* \tag{36}$$

The total power entering the n-port network from all external sources is therefore given by the sum of the powers entering each port [5].-

$$P_{tot} = \sum_{i=1}^{n} P_i = \sum_{i=1}^{n} (a_i a_i^* - b_i b_i^*) \tag{37}$$

Equation (37) can be rewritten in matrix form as follows.-

$$P_{tot} = [a]^\dagger [a] - [b]^\dagger [b] \tag{38}$$

Where $[M]^\dagger$ denotes the conjugate transpose of the matrix M. Employing the fact that $[b] = [S][a]$, equation (38) can be written as :

$$P_{tot} = [a]^\dagger [a] - [a]^\dagger [S]^\dagger [S][a] \tag{39}$$

Activity and Passivity

Equation (39) can be written as :

$$P_{tot} = [a]^\dagger \left[[I] - [S]^\dagger [S]\right] [a] \tag{40}$$

The matrix $\left[[I] - [S]^\dagger [S]\right]$ is known as the *dissipation matrix* and is represented as $[Q]$. The total power flow into the n-port network given by (40) can thus be written in the concise form:-

$$P_{tot} = [a]^\dagger [Q][a] \tag{41}$$

For an active network the net flow of signal power *into* the network is negative (i.e. $P_{tot} < 0$), meaning that there is a net positive flow of signal power out of the network. By contrast, for a passive network the net flow of signal power into the network is positive or zero, i.e. $P_{tot} \geq 0$
The activity and passivity criteria can be stated in terms of the S-parameters of the network. For the passive network, equation (41) becomes.:

$$[a]^\dagger [Q][a] \geq 0 \tag{42}$$

Activity and Passivity

Equation (42) means that the dissipation matrix of a passive network must be Positive Definite (PD) or Positive Semi-Definite (PSD). The necessary and sufficient condition for a Hermitian matrix to be Positive Definite is that its determinant and that of its principal minors be non-negative [5].

This can be illustrated by considering the two-port scattering matrix of equation (13). The dissipation matrix for the two-port network is given by.:

$$[Q] = \begin{bmatrix} \left(1 - |S_{11}|^2 - |S_{21}|^2\right) & -\left(S_{11}^* S_{12} + S_{21}^* S_{22}\right) \\ -\left(S_{11} S_{12}^* + S_{21} S_{22}^*\right) & \left(1 - |S_{12}|^2 - |S_{22}|^2\right) \end{bmatrix} \tag{43}$$

For the principal minors of the dissipation matrix to be non-negative, the following two-conditions must be simultaneously satisfied:

$$1 - |S_{11}|^2 - |S_{21}|^2 \geq 0 \tag{44}$$

$$|Q|^2 \geq 0 \tag{45}$$

Expanding the determinant of [Q] and simplifying leads to a restatement of equation (45):

$$\left(1 - |S_{11}|^2 - |S_{12}|^2 - |S_{21}|^2 - |S_{22}|^2\right) + |\Delta|^2 \geq 0 \tag{46}$$

Where

$$\Delta = S_{11}S_{22} - S_{12}S_{21}$$

Activity and Passivity

Replacing (45) by (46), the passivity criteria can be stated thus:

$$|S_{11}|^2 + |S_{21}|^2 \leq 1 \tag{47}$$

$$\frac{|S_{11}|^2 + |S_{12}|^2 + |S_{21}|^2 + |S_{22}|^2}{1 + |\Delta|^2} \geq 1 \tag{48}$$

The passivity criteria must be invariant when input and output are interchanged therefore a further condition can be stated based on (47) :

$$|S_{22}|^2 + |S_{12}|^2 \geq 1 \tag{49}$$

If any of the conditions (47) to (49) are violated then the device is active. It is possible, however, for condition (47) or condition (49) to be violated even though the magnitudes of the individual scattering parameters are less than unity. But furthermore, condition (48) can be violated even though the remaining two conditions are satisfied, meaning that this condition is not redundant.

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Signal flow graphs

A signal flow graph is a pictorial representation of a system of simultaneous equations. The technique was originally developed for use in control theory, but was later applied to S-parameter circuit analysis by the likes of Kuhn [9].

Once a network has been described in terms of a signal flow graph, some basic rules can be applied to analyse and simplify the network. It is often quicker to use the signal flow graph technique than to use algebraic manipulation, although the two approaches yield identical results, as will be demonstrated here.

A signal flow graph is made up of *Nodes* and *Branches*, which are defined as follows:

- Nodes:
  1. Each port, i, of a microwave network has two nodes, a_i and b_i.
  2. Node a_i is identified with a wave entering port i, while node b_i is identified with a wave reflected from port i.
  3. The voltage at a node is equal to the sum of all signals entering that node.
- Branches :
  1. A branch is a directed path between two nodes, representing signal flow from one node to another.
  2. Every branch has an associated coefficient (i.e. an S-parameter or reflection coefficient).

Signal flow graphs

A signal flow graph representation of power wave flows in a two-port network is shown in figure 5. In this case, a wave of amplitude a_1 incident at port 1 is split and proceeds as follows :

1. A portion of the incident power wave passes through S_{11} and comes out at port 1 as a reflected wave
2. The remainder of the power wave gets transmitted through S_{21} and emerges from node b_2.
3. If a load with a non-zero reflection coefficient is connected at port 2, the wave emerging from node b_2 will be partly reflected at the load and will re-enter the two-port network at node a_2.
4. Part of the reflected wave entering at node a_2 will be reflected back out of port 2 via S_{22}.
5. The remainder will be transmitted out from port 1 through S_{12}.

$$
\begin{tikzcd}
\arrow[r] & a_1 \arrow[r, "S_{21}"] \arrow[d, "S_{11}"'] & b_2 \arrow[r] & {} \\
{} & b_1 \arrow[l] & a_2 \arrow[l, "S_{12}"] \arrow[u, "S_{22}"'] & \arrow[l]
\end{tikzcd}
$$

Figure 5 : Signal flow graph representation of a two-port network

Decomposition of signal flow graphs

Rule 1 (Series Rule) :

Two branches, whose common node has only one incoming and one outgoing wave (branches in series), may be combined to form a single branch whose coefficient is the product of the coefficients of the original branches. Figure 6 illustrates this rule. The signal flow graph in figure 6(b) is equivalent to figure 6(a).

$$a_1 \xrightarrow{\quad S_{21} \quad} a_2 \xrightarrow{\quad S_{32} \quad} a_3$$

(a)

$$a_1 \xrightarrow{\qquad\quad S_{21}S_{32} \qquad\quad} a_3$$

(b)

Figure 6 : Signal flow graph: Rule 1

Decomposition of signal flow graphs

Rule 2 (Parallel Rule) :
Two branches from one common node to another common node (branches in parallel) may be combined into a single branch whose coefficient is the sum of the coefficients of the original branches. Figure 7 illustrates this rule. The signal flow graph in figure 7(b) is equivalent to figure 7(a).

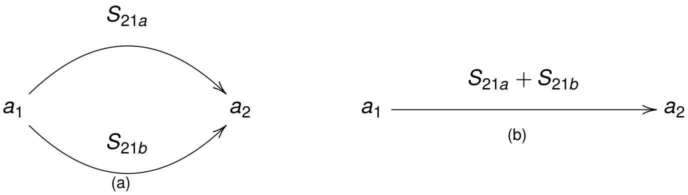

Figure 7 : Signal flow graph: Rule 2

Decomposition of signal flow graphs

Rule 3 (Self-Loop Rule) :
When a node has a self-loop (a branch that begins and ends at the same node) of coefficient S_{xx}, the self-loop can be eliminated by multiplying coefficients of the branches feeding that node by $1/(1 - S_{xx})$. This can be proven with reference to figure 8 as follows :

$$
\begin{aligned}
a_2 &= S_{21} a_1 + S_{22} a_2 \\
a_2(1 - S_{22}) &= S_{21} a_1 \\
\therefore a_2 &= \frac{S_{21}}{(1 - S_{22})} a_1
\end{aligned}
$$

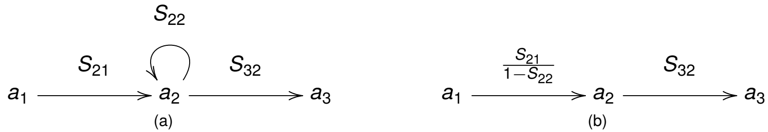

Figure 8 : Signal flow graph: Rule 3

Decomposition of signal flow graphs

Rule 4 (Splitting Rule) :
A node may be split into two separate nodes as long as the resulting flow graph contains, once and only once, each combination of separate (not self loops) input and output branches that connect to the original node. Figure 9 illustrates this rule. The signal flow graph in figure 9(b) is equivalent to figure 9(a).

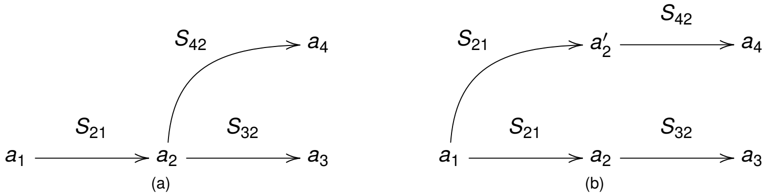

Figure 9 : Signal flow graph: Rule 4

Decomposition of signal flow graphs

Mason's rule :

Mason's Rule, or 'Mason's non-touching loop rule', is a useful short-cut that allows us to calculate the transfer function for a signal flow graph by 'inspection'.

The rule will be explained with reference to figure 10, which represents a two-port with load, Γ_L, and being driven by a source with reflection coefficient, Γ_S, included.

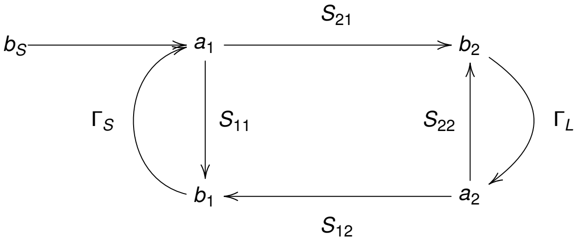

Figure 10 : Two-port network with load (illustration of Mason's rule)

The network of figure 10 has basically only one independent variable, namely b_S, the power wave emanating from the source.

Mason's Rule

We can analyse figure 10 in terms of *paths* and *loops*, which are defined as follows :

- A path is a series of directed lines followed in sequence and in the same direction in such a way that no node is touched more than once. The value of the path is the product of all the coefficients encountered in the process of traversing the path. There are three paths in figure 10, namely :
  1. The path from b_S to b_2 having the value S_{21}.
  2. The path from b_S to b_1 having the value S_{11}.
  3. The path from b_S to b_1 having the value $S_{21}\Gamma_L S_{12}$.
- A first order loop is a series of directed lines coming to a closure when followed in sequence and in the same direction with no node passed more than once. The value of the loop is the product of all coefficients encountered in the process of traversing the loop. There are three first order loops in figure 10, namely :
  1. The loop a_1 - b_1 having the value $\Gamma_S S_{11}$.
  2. The loop b_2 - a_2 having the value $S_{22}\Gamma_S$.
  3. The loop a_1 - b_2 - a_2 - b_1 having the value $S_{21}\Gamma_L S_{12}\Gamma_S$.
- A second order loop is the product of any two first order loops which do not touch at any point. There is one second order loop in figure 10, namely $\Gamma_S S_{11} S_{22}\Gamma_L$.
- A third order loop is the product of any three first order loops which do not touch. There are no third order loops in figure 10.
- An n^{th} order loop in general, is the product of any n first order loops which do not touch.

Mason's Rule

Mason's rule can be expressed symbolically as shown in (50)

$$T = \frac{P_1[1 - \sum \mathcal{L}(1)^{(1)} + \sum \mathcal{L}(2)^{(1)} - \ldots] + P_2[1 - \sum \mathcal{L}(1)^{(2)} + \ldots]}{1 - \sum \mathcal{L}(1) + \sum \mathcal{L}(2) - \sum \mathcal{L}(3) + \ldots} \tag{50}$$

- Each P_i in (50) denotes a path which can be followed from the independent variable node to the node whose value we wish to determine. The notation $\sum \mathcal{L}(1)$ denotes the sum over all first order loops.
- The notation $\sum \mathcal{L}(2)$ denotes the sum over all second order loops, and so on. The notation $\sum \mathcal{L}(1)^{(1)}$ denotes the sum of all the first order loops which do not touch P_1 at any point.
- The notation $\sum \mathcal{L}(2)^{(1)}$ denotes the sum of all the second order loops which do not touch P_1 at any point, and so on. In both these cases, the superscript (1) denotes path 1. By extension the superscript (2) denotes path 2, etc.

Mason's Rule

To avoid confusion in the application of Mason's rule, it helps to list out all the paths and n^{th} order loops for the circuit in question. We can then apply these to (50) systematically. Let's say, for example, we wanted to determine the transfer function $T = b_2/b_S$ for the circuit of figure 10. We would therefore compile the following list :

- Paths: In this example, there is only one path from b_S to b_2, which we shall call P_1. The value of this path is S_{21}.
- First order loops: We can list these as follows:
  1. $\Gamma_S S_{11}$.
  2. $S_{22}\Gamma_S$.
  3. $S_{21}\Gamma_L S_{12}\Gamma_S$.
- Second order loops: These are the product of two non-touching first-order loops. For instance, since loops $S_{11}\Gamma_S$ and $S_{22}\Gamma_L$ do not touch, their product is the one and only second order loop, i.e. $S_{11}\Gamma_S S_{22}\Gamma_L$

Mason's Rule

We can now write the numerator of (50) as :

$$P_1[1 - \sum \mathcal{L}(1)^{(1)} + \sum \mathcal{L}(2)^{(1)} - \ldots] + P_2[1 - \sum \mathcal{L}(1)^{(2)} + \ldots] = S_{21}(1 - 0) \quad (51)$$

and the denominator of (50) as :

$$1 - \sum \mathcal{L}(1) + \sum \mathcal{L}(2) = 1 - \Gamma_S S_{11} - S_{22}\Gamma_S - S_{21}\Gamma_L S_{12}\Gamma_S + S_{11}\Gamma_S S_{22}\Gamma_L \quad (52)$$

Now applying (50), we can write the transfer function $T = b_2/b_S$ as :

$$T = \frac{S_{21}}{1 - \Gamma_S S_{11} - S_{22}\Gamma_L - S_{21}\Gamma_L S_{12}\Gamma_S + S_{11}\Gamma_S S_{22}\Gamma_L} \quad (53)$$

The denominator of equation (53) is in the form of $1 - x - y + xy$ which can be factored as $(1 - x)(1 - y)$. We can therefore rewrite (53) as :

$$T = \frac{S_{21}}{(1 - \Gamma_S S_{11})(1 - S_{22}\Gamma_L) - S_{21}\Gamma_L S_{12}\Gamma_S} \quad (54)$$

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Scattering Transfer Parameters

It is useful to be able to calculate the S-parameters of two or more two-port networks in cascade. If we rearrange the power wave expression to make the input power waves the independent variables, we get a set of two-port parameters which are referred to as the *Transfer Scattering parameters* or *T-parameters*.
Referring to the two-port shown in figure 2, if we choose b_1 and a_1 as the independent variables and b_2 and a_2 as the dependent variables we have:

$$\left.\begin{aligned} a_1 &= T_{11} b_2 + T_{12} a_2 \\ b_1 &= T_{21} b_2 + T_{22} a_2 \end{aligned}\right\} \tag{55}$$

Or, in matrix notation :

$$\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} b_2 \\ a_2 \end{bmatrix} \tag{56}$$

Where T_{ij} are known as the *transfer scattering parameters*, or T-parameters[2]. Equation (56) relates the power waves at one-port to the power waves at the other port.

Scattering Transfer Parameters

The relationship between the S-parameters and the T-parameters may be found by comparing equations (56) and (15) which results in:

$$\left.\begin{aligned} T_{11} &= \frac{S_{12}S_{21} - S_{11}S_{22}}{S_{21}} = \frac{-\Delta}{S_{21}} \\ T_{12} &= \frac{S_{11}}{S_{21}} \\ T_{21} &= -\frac{S_{22}}{S_{21}} \\ T_{22} &= \frac{1}{S_{21}} \end{aligned}\right\} \tag{57}$$

Where :

$$\Delta = S_{11}S_{22} - S_{12}S_{21} \tag{58}$$

Scattering Transfer Parameters

Rearrangement of equations (57) gives the S-parameters in terms of the T-parameters : -

$$\left.\begin{aligned} S_{11} &= \frac{T_{12}}{T_{22}} \\ S_{12} &= \frac{T_{11}T_{22} - T_{12}T_{21}}{T_{22}} = \frac{\Delta_T}{T_{22}} \\ S_{21} &= \frac{1}{T_{22}} \\ S_{22} &= -\frac{T_{21}}{T_{22}} \end{aligned}\right\} \tag{59}$$

Where :

$$\Delta_T = T_{11}T_{22} - T_{21}T_{12} \tag{60}$$

Scattering Transfer Parameters

The primary application of T-parameters, as just mentioned, is in calculating the overall parameters of two cascaded two-ports, as illustrated in figure 12. The T-parameters can therefore be considered as the power wave equivalent of the ABCD immittance parameters introduced in section **??**

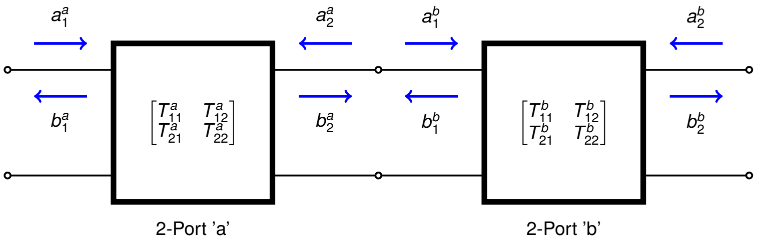

Figure 11 : Cascaded two-ports represented by T-parameters

For two-port 'a' we have:

$$\begin{bmatrix} a_1^a \\ b_1^a \end{bmatrix} = \begin{bmatrix} T_{11}^a & T_{12}^a \\ T_{21}^a & T_{22}^a \end{bmatrix} \begin{bmatrix} b_2^a \\ a_2^a \end{bmatrix} \tag{61}$$

For two-port 'b' we have:

$$\begin{bmatrix} a_1^b \\ b_1^b \end{bmatrix} = \begin{bmatrix} T_{11}^b & T_{12}^b \\ T_{21}^b & T_{22}^b \end{bmatrix} \begin{bmatrix} b_2^b \\ a_2^b \end{bmatrix} \tag{62}$$

Scattering Transfer Parameters

Therefore, we can combine equations (61) and (62) to obtain:

$$\begin{bmatrix} a_1^a \\ b_1^a \end{bmatrix} = \begin{bmatrix} T_{11}^a & T_{12}^a \\ T_{21}^a & T_{22}^a \end{bmatrix} \begin{bmatrix} T_{11}^b & T_{12}^b \\ T_{21}^b & T_{22}^b \end{bmatrix} \begin{bmatrix} b_2^b \\ a_2^b \end{bmatrix} \tag{63}$$

Or:

$$\begin{bmatrix} a_1^a \\ b_1^a \end{bmatrix} = \begin{bmatrix} T_{11}^{ab} & T_{12}^{ab} \\ T_{21}^{ab} & T_{22}^{ab} \end{bmatrix} \begin{bmatrix} b_2^b \\ a_2^b \end{bmatrix} \tag{64}$$

Where:

$$\begin{bmatrix} T_{11}^{ab} & T_{12}^{ab} \\ T_{21}^{ab} & T_{22}^{ab} \end{bmatrix} = \begin{bmatrix} T_{11}^a & T_{12}^a \\ T_{21}^a & T_{22}^a \end{bmatrix} \begin{bmatrix} T_{11}^b & T_{12}^b \\ T_{21}^b & T_{22}^b \end{bmatrix} \tag{65}$$

For the case of the two cascaded two-ports in figure 12 we then have, from (65) :

$$\left.\begin{aligned} T_{11}^{ab} &= T_{11}^a T_{11}^b + T_{12}^a T_{21}^b \\ T_{12}^{ab} &= T_{11}^a T_{12}^b + T_{12}^a T_{22}^b \\ T_{21}^{ab} &= T_{21}^a T_{11}^b + T_{22}^a T_{21}^b \\ T_{22}^{ab} &= T_{21}^a T_{12}^b + T_{22}^a T_{22}^b \end{aligned}\right\} \tag{66}$$

S-parameters of cascaded two-ports

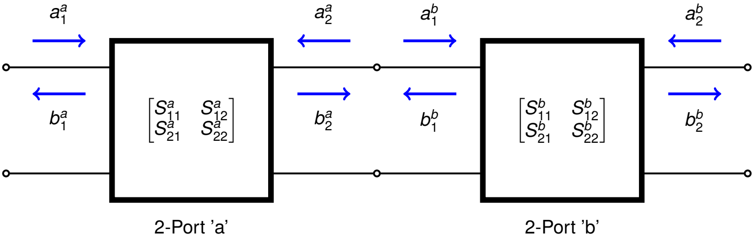

Figure 12 : Cascaded two-ports represented by T-parameters

In the special case of a cascade of two two-port networks, characterised by the two port matrices $[S^a]$ and $[S^b]$, respectively, as shown in figure 12, we can write the overall S-matrix of the cascade as :

$$\begin{bmatrix} S_{11}^{ab} & S_{12}^{ab} \\ S_{21}^{ab} & S_{22}^{ab} \end{bmatrix} = \frac{1}{1 - S_{22}^a S_{11}^b} \begin{bmatrix} (S_{11}^a - S_{11}^b \Delta^a) & S_{12}^a S_{12}^b \\ S_{21}^b S_{21}^a & (S_{22}^b - S_{22}^a \Delta^b) \end{bmatrix} \tag{67}$$

Where $\Delta^a = S_{11}^a S_{22}^a - S_{12}^a S_{21}^a$ and $\Delta^b = S_{11}^b S_{22}^b - S_{12}^b S_{21}^b$

Relationship between S-parameters and immittance parameter

A set of equations relating the S-parameters to immitance parameters can derived using straightforward matrix algebra. Taking the Z-parameters as an example, we can write :

$$[V] = [Z][I] \tag{68}$$

In the case of a two-port network, $[Z]$ can be written as :

$$[Z] = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \tag{69}$$

with the voltage and current vectors being :

$$[V] = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \tag{70}$$

$$[I] = \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \tag{71}$$

Relationship between S-parameters and immittance parameter

The instantaneous voltages or currents at the ports is the sum of incident and reflected voltages or currents. We can therefore write (68) in terms of incident and reflected voltage and current matrices as follows :

$$[V^+] + [V^-] = [Z]\left([I^+] - [I^-]\right) \tag{72}$$

Where the respective voltage and current waves are related by :

$$[V^+] = [Z_o][I^+] \tag{73}$$

and

$$[V^-] = [Z_o][I^-] \tag{74}$$

Substituting (73) and (74) into (72) we can write :

$$([Z] + [Z_o])\,[I^-] = ([Z] - [Z_o])\,[I^+] \tag{75}$$

The matrix ratios of reflected quantities, $[I^-]$ and $[V^-]$ to the respective incident quantities $[I^+]$ and $[V^+]$ are essentially the S-matrix of the two-port. In other words we can write, in matrix notation :

$$[S] = [V^+]^{-1}[V^-] = [I^+]^{-1}[I^-] \tag{76}$$

Combining (75) and (76), we can now define the S-matrix purely in terms of $[Z]$ and $[Z_o]$:

$$[S] = ([Z] + [Z_o])^{-1}\,([Z] - [Z_o]) \tag{77}$$

Relationship between S-parameters and immittance parameter

In the two-port case we can expand the matrix relationship (77) into individual expressions for each of the two-port S-parameters, as follows :

$$S_{11} = \frac{(Z_{11} - Z_o)(Z_{22} + Z_o) - Z_{12}Z_{21}}{(Z_{11} + Z_o)(Z_{22} + Z_o) - Z_{12}Z_{21}}$$

$$S_{12} = \frac{2Z_{12}Z_o}{(Z_{11} + Z_o)(Z_{22} + Z_o) - Z_{12}Z_{21}}$$

$$S_{21} = \frac{2Z_{21}Z_o}{(Z_{11} + Z_o)(Z_{22} + Z_o) - Z_{12}Z_{21}}$$

$$S_{22} = \frac{(Z_{11} + Z_o)(Z_{22} - Z_o) - Z_{12}Z_{21}}{(Z_{11} + Z_o)(Z_{22} + Z_o) - Z_{12}Z_{21}}$$

A similar matrix analysis to the above could be carried out in terms of the Y-parameters, h-parameters and the ABCD-parameters.

Table of Contents

The Network Analyser

There are basically two categories of network analyser, namely[4, 6]:

1. Scalar Network Analyzer (SNA) - measures amplitude ratios only.
2. Vector Network Analyzer (VNA) - measures both amplitude and phase ratios.

- In general, a VNA is simply referred to as a "Network Analyser" by default, as these are the most common type of network analyser in use today.
- The term *Automatic Network Analyzer* (ANA) is also used to describe a VNA which is somehow programmable by means of a built in computer, that is to say all of them these days.
- Finally, Keysight Technologies (formerly Hewlett Packard, and later Agilent Technologies Inc.) now classify their ANA product range into 'high-Performance Network Analysers' (PNA) and "Economy Network Analysers" (ENA) categories.

The Network Analyser

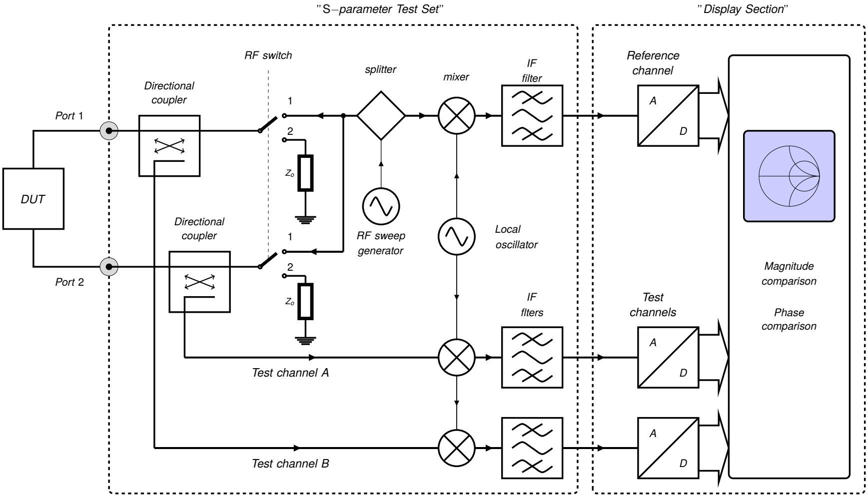

Figure 13 : Vector Network Analyser: conceptual block diagram

VNA measurement procedure

In order to measure S_{11} and S_{21}, the RF switch in figure 13 is set to position 1 so that the DUT is driven from port 1. The reflected power wave b_1 is sampled by the port 1 directional coupler and the transmitted power wave b_2 is sampled by the port 2 directional coupler. The reference channel signal represents the incident power wave, a_1. The digital signal processor in the display section calculates the following ratios :

$$S_{11} = \frac{|b_1|}{|a_1|}\angle\Phi_{11} \tag{78}$$

$$S_{21} = \frac{|b_2|}{|a_1|}\angle\Phi_{21} \tag{79}$$

In order to measure S_{12} and S_{22}, the RF switch is set to position 2. The DUT is then driven from port 2. The reflected power wave b_2 is sampled by the port 2 directional coupler and the transmitted power wave b_1 is sampled by the port 1 directional coupler. This time, the reference channel signal represents the incident power wave, a_2. The digital signal processor in the display section now calculates the following ratios :

$$S_{22} = \frac{|b_2|}{|a_2|}\angle\Phi_{22} \tag{80}$$

$$S_{12} = \frac{|b_1|}{|a_2|}\angle\Phi_{12} \tag{81}$$

The above switching of the RF switch is under control of the main computer in the display section. This allows the VNA to automatically measure all four parameters without the user having to physically reverse the port connections to the DUT.

Network Analyser Error Correction

The main sources of error in S-parameter measurements using a network analyser are [1, 11, 7]:-

(i) Errors due to finite directivity of the couplers (directivity error).

(ii) Errors due to the source not being a perfect Z_o (source mismatch error).

(iii) Errors due to the reference and test channels not having identical amplitude and phase response with frequency (frequency tracking error).

(iv) Crosstalk between reference and test channels.

The error effects (iii) and (iv) are generally grouped together under the term "frequency response error".
In order to take these error effects into account, it is necessary to have a mathematical model which relates the true device characteristics to those measured.

One-port error model (the '3-term' model)

We can consider all the three sources of error associated with a one-port measurement to be combined together in the form of an 'error two-port' inserted between a perfect (error less) test set and the true value of the load we wish to measure. Consider that we wish to measure the actual value of S_{11} of the DUT, which we shall denote S_{11}^a, whereas the actual measured value we will denote S_{11}^m. What stands between the actual and measured values of S_{11} is the *error two-port*, represented by means of a signal flow graph[3, 9] in figure 14.

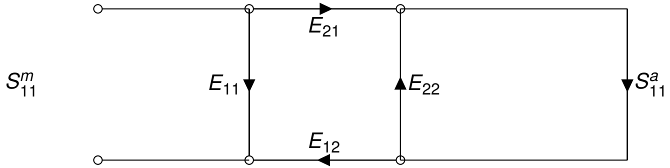

Figure 14 : One-port error model

The relationship between the error terms E_{ij} of figure 14 and the sources of error listed at the beginning of this section are as follows:

- E_{11} = Combined effect of coupler directivity error and connector and cable loss.
- $E_{12} \cdot E_{21}$ = Combined effect of frequency response and cable losses.
- E_{22} = Source mismatch error.

One-port error model (the '3-term' model)

Relationships between input and output reflection coefficients of a two-port with arbitrary terminations have been derived in section **??**, as follows :

$$\Gamma_{in} = S_{11} + \frac{S_{21}S_{12}\Gamma_L}{1 - S_{22}\Gamma_L} \tag{20}$$

$$\Gamma_{out} = S_{22} + \frac{S_{21}S_{12}\Gamma_S}{1 - S_{11}\Gamma_S} \tag{22}$$

Where:
Γ_s = Source reflection coefficient.
Γ_L = load reflection coefficient.
Γ_{in} = Input reflection coefficient of two-port.
Γ_{out} = Output reflection coefficient of two-port.
Applying equation (20) to figure 14 we can relate the true value of the load we are trying to measure (S^a_{11}) to the measured value (S^m_{11}) as follows :

$$S^m_{11} = E_{11} + \frac{S^a_{11}(E_{21}E_{12})}{1 - S^a_{11}E_{22}} \tag{82}$$

Rearranging equation (82) we obtain S^a_{11} in terms of S^m_{11} and the error terms as follows :

$$S^a_{11} = \frac{S^m_{11} - E_{11}}{E_{22}(S^m_{11} - E_{11}) + E_{12}E_{21}} \tag{83}$$

Calibration procedure : Step 1 (matched load)

Measure S_{11}^m for a "perfect" matched load, as shown in figure 15

$S_{11}^m = E_{11}$

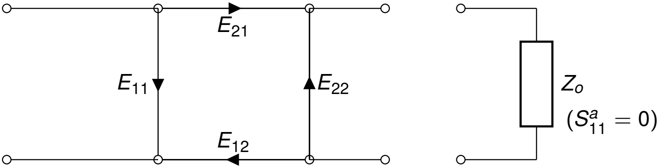

Figure 15 : One-port error model : matched load

In this case $S_{11}^m = 0$ and equation (82) becomes:

$$S_{11}^m = E_{11} \tag{84}$$

The computer therefore stores this measured value of S_{11}^m as the error parameter E_{11}.

Calibration procedure : Step 2 (short)

Measure S_{11}^m for a short circuit termination, as shown in figure 16

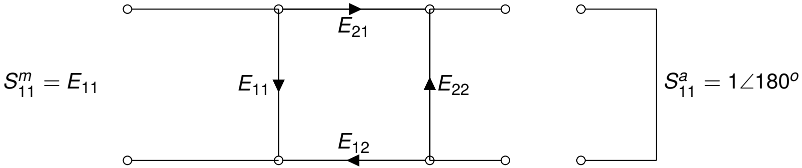

Figure 16 : One-port error model : short circuit termination

In this case $S_{11a} = 1\angle 180^o$ and equation (82) becomes:

$$S_{11}^m = E_{11} - \frac{(E_{21}E_{12})}{1 + E_{22}} \tag{85}$$

Calibration procedure : Step 3 (open)

Measure S_{11}^m for an open circuit termination, as shown in figure 17:

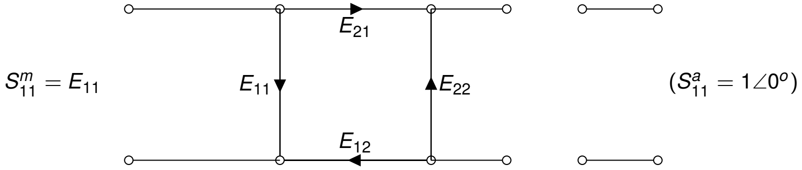

Figure 17 : One-port error model : open circuit termination

In this case $S_{11a} = 1\angle 0^o$ and equation (82) becomes:

$$S_{11}^m = E_{11} + \frac{(E_{21}E_{12})}{1 - E_{22}} \tag{86}$$

The computer simultaneously solves equations (85) and (86) to obtain E_{22} and the product $E_{12} \cdot E_{21}$. These three complex error terms (E_{11}, E_{22} and $E_{21} \cdot E_{12}$) are stored for each frequency and used to correct each S_{11}^m measurement made.

Two-port error model (the '12-term' model)

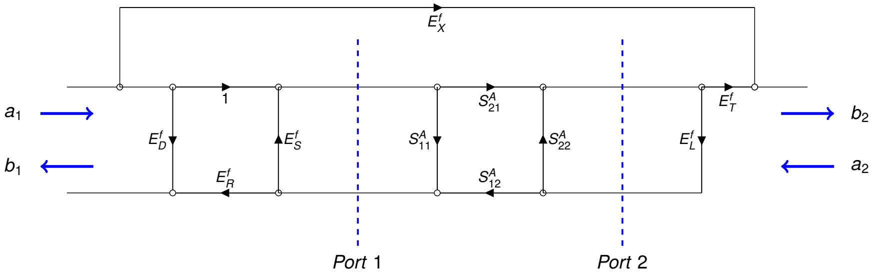

Figure 18 : Two-port error model : forward

E_D^f =Forward directivity error.

E_S^f =Forward source mismatch error.

E_R^f =Forward reflection tracking error.

E_L^f =Forward load match error.

E_T^f =Forward transmission tracking error.

E_X^f =Forward isolation error.

Two-port error model (the '12-term' model)

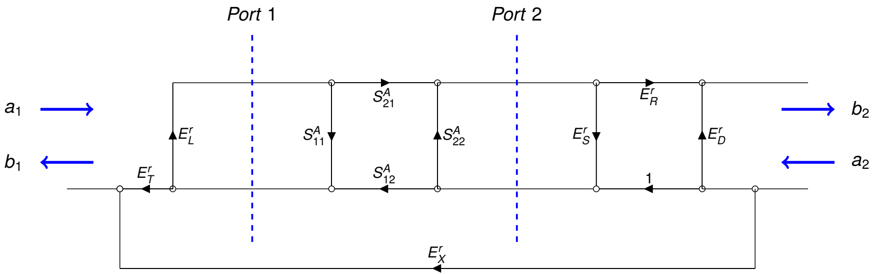

Figure 19 : Two-port error model : reverse

E_D^r =Reverse directivity error.

E_S^r =Reverse source mismatch error.

E_R^r =Reverse reflection tracking error.

E_L^r =Reverse load match error.

E_T^r =Reverse transmission tracking error.

E_X^r =Reverse isolation error.

Two-port error model (the '12-term' model)

Each actual S-parameter is a function of all four measured S-parameters and may be calculated from the following relationships[8]:

$$S_{11}^{a} = \frac{\left(\frac{S_{11}^{m} - E_{D}^{f}}{E_{R}^{f}}\right)\left(1 + \frac{S_{22}^{m} - E_{D}^{r}}{E_{R}^{r}} E_{S}^{r}\right) - E_{L}^{f}\left(\frac{S_{21}^{m} - E_{X}^{f}}{E_{T}^{f}}\right)\left(\frac{S_{12}^{m} - E_{X}^{r}}{E_{T}^{r}}\right)}{\left(1 + \frac{S_{11}^{m} - E_{D}^{f}}{E_{R}^{f}} E_{S}^{f}\right)\left(1 + \frac{S_{22}^{m} - E_{D}^{r}}{E_{R}^{r}} E_{S}^{r}\right) - E_{L}^{r} E_{L}^{f}\left(\frac{S_{21}^{m} - E_{X}^{f}}{E_{T}^{f}}\right)\left(\frac{S_{12}^{m} - E_{X}^{r}}{E_{T}^{r}}\right)} \tag{87}$$

$$S_{21}^{a} = \frac{\left(\frac{S_{21}^{m} - E_{X}^{f}}{E_{T}^{f}}\right)\left[1 + \frac{S_{22}^{m} - E_{D}^{r}}{E_{R}^{r}}\left(E_{S}^{r} - E_{L}^{f}\right)\right]}{\left(1 + \frac{S_{11}^{m} - E_{D}^{f}}{E_{R}^{f}} E_{S}^{f}\right)\left(1 + \frac{S_{22}^{m} - E_{D}^{r}}{E_{R}^{r}} E_{S}^{r}\right) - E_{L}^{r} E_{L}^{f}\left(\frac{S_{21}^{m} - E_{X}^{f}}{E_{T}^{f}}\right)\left(\frac{S_{12}^{m} - E_{X}^{r}}{E_{T}^{r}}\right)} \tag{88}$$

Two-port error model (the '12-term' model)

$$S^a_{12} = \frac{\left(\dfrac{S^m_{12} - E^r_X}{E^r_T}\right)\left[1 + \dfrac{S^m_{11} - E^f_D}{E^f_R}\left(E^f_S - E^r_L\right)\right]}{\left(1 + \dfrac{S^m_{11} - E^f_D}{E^f_R} E^f_S\right)\left(1 + \dfrac{S^m_{22} - E^r_D}{E^r_R} E^r_S\right) - E^r_L E^f_L \left(\dfrac{S^m_{21} - E^f_X}{E^f_T}\right)\left(\dfrac{S^m_{12} - E^r_X}{E^r_T}\right)} \tag{89}$$

$$S^a_{22} = \frac{\left(\dfrac{S^m_{22} - E^r_D}{E^f_R}\right)\left(1 + \dfrac{S^m_{11} - E^f_D}{E^f_R} E^r_S\right) - E^r_L \left(\dfrac{S^m_{21} - E^f_X}{E^f_T}\right)\left(\dfrac{S^m_{12} - E^r_X}{E^r_T}\right)}{\left(1 + \dfrac{S^m_{11} - E^f_D}{E^f_R} E^f_S\right)\left(1 + \dfrac{S^m_{22} - E^r_D}{E^r_R} E^r_S\right) - E^r_L E^f_L \left(\dfrac{S^m_{21} - E^f_X}{E^f_T}\right)\left(\dfrac{S^m_{12} - E^r_X}{E^r_T}\right)} \tag{90}$$

References

S F Adam.
Microwave theory and applications.
Prentice-Hall Inc., Englewood cliffs, N.J., 1969.

Agilent Technologies Inc.
De-embedding and embedding S-Parameter networks using a vector network analyzer.
Application Note 1364-1, 2004.

G Bodway.
Circuit Design and Characterisation of transistors by means of three-port scattering parameters.
Microwave Journal, 11(5):4–27, May 1968.

G.H. Bryant and Institution of Electrical Engineers.
Principles of Microwave Measurements.
IEE electrical measurement series. P. Peregrinus Limited, 1993.

H J Carlin and A B Giordano.
Network theory: an introduction to reciprocal and non-reciprocal circuits.
Prentice-Hall Inc., Englewood cliffs, N.J, 1964.

R.J. Collier and A.D. Skinner.
Microwave Measurements, 3rd Edition.
IET electrical measurement series. Institution of Engineering and Technology, 2007.

J Fitzpatrick.
Error models for systems measurement.
Microwave Journal, 22(5):63–66, May 1978.