Lecture 2 - Transmission Line Theory

Microwave Active Circuit Analysis and Design

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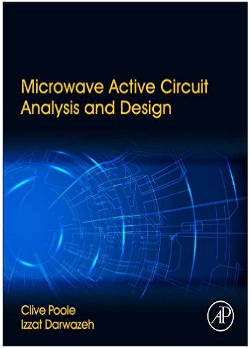
Academic Press Inc.



Intended Learning Outcomes

- Knowledge
  - Understand that electrical energy travels at a finite speed in any medium, and the implications of this.
  - Understand the behaviour of lossy versus lossless transmission lines.
  - Understand power flows on a transmission line and the effect of discontinuities.
- Skills
  - Be able to determine the location of a discontinuity in a transmission line using time domain refractometry.
  - Be able to apply the telegrapher's equations in a design context.
  - Be able to calculate the reflection coefficient, standing wave ratio of a transmission line of known characteristic impedance with an arbitrary load.
  - Be able to calculate the input impedance of a transmission line of arbitrary physical length, and terminating impedance.
  - Be able to determine the impedance of a load given only the voltage standing wave ratio and the location of voltage maxima and minima on a line.

Table of Contents

Propagation and reflection on a transmission line

Let us consider a simple lossless transmission line, which could be simply a pair of parallel wires, terminated in a resistive load and connected to a DC source, such as a battery having a finite internal resistance, R_S.

We will add one minor refinement to this simple picture in the form of a changeover switch at the source, as shown in figure 1, which enables us to disconnect the battery by moving the switch from position A to position B, but ensures that the line continues to see the same source impedance, R_S, whether the battery is connected or not.

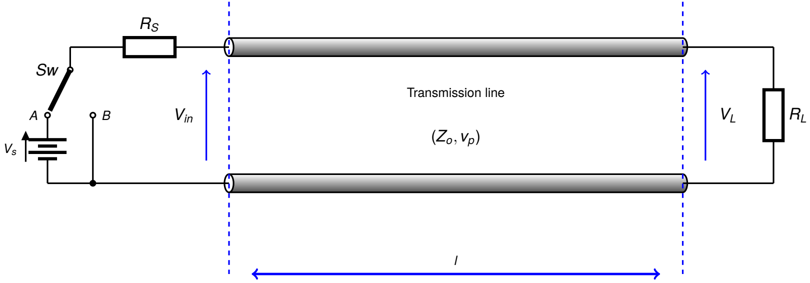


Figure 1 : Simple transmission line system, DC driven

Propagation and reflection on a transmission line

Consider the switch in figure 1 is initially in position B and is then instantaneously moved to position A at time we shall define as $t = 0$.

Once the circuit reaches a steady state (i.e. at $t >> 0$), V_{in} is determined by the potential divider formed by the source resistance, R_s and the load resistance R_L as follows :

$$V_{in} = V_s \left(\frac{R_L}{R_s + R_L} \right) \tag{1}$$

But electrical energy in the line travels at a finite speed. At time $t = 0$, therefore, the source has no way of "knowing" the nature of the termination at the other end of the line. At $t = 0$, the line may as well be infinitely long as far as the determination of V_{in} is concerned.

We do know, however, that a finite current does start to flow into the line at $t = 0$, otherwise we would never reach the steady state condition. The fact that a finite current is flowing at $t = 0$ suggests that the source is "seeing" a finite impedance at the input of the line, and a corresponding finite voltage, V_{in}, is being developed across it. Since we are talking about $t = 0$, i.e. before the current has had time to traverse the full physical length of the line, we must conclude that this mystery line impedance, which we shall call the *characteristic impedance*, is an inherent property of this particular line and is independent of both the physical length of the line and the value of the load.

Propagation and reflection on a transmission line

If we assign Z_o to this characteristic impedance, we can now write the voltage, V_{in}, at time $t = 0$ as follows:

$$V_{in} = V_s \left(\frac{Z_o}{R_s + Z_o} \right) \tag{2}$$

The characteristic impedance is the impedance of an infinitely long transmission line of specific physical characteristics. This can be understood by considering, once again, the fact that at time $t = 0$ any transmission line of non-zero length will appear to be infinitely long, as far as the source is concerned.

So, as time progresses, from $t = 0$ onwards, a voltage wave, of amplitude V_{in}, will propagate down the line towards the load. We will refer to this forward-propagating voltage wave as the *incident* voltage, V_i. The corresponding incident current, I_i, propagating down the line in a direction away from the source is given by :

$$I_i = \frac{V_i}{Z_o} \tag{3}$$

Propagation and reflection on a transmission line

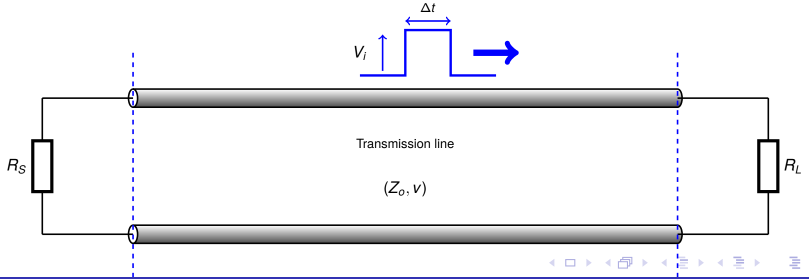
Let us now assume that the switch, having been moved to position A at time $t = 0$, is moved back to position B a short time later at a time $t = \Delta t$. We choose Δt to be much less than the time, τ, it takes for electrical energy to propagate down the full length of the line from source to load, i.e :

$$\Delta t << \tau \tag{4}$$

Where :

$$\tau = \frac{l}{v_p} \tag{5}$$

Where v_p is the velocity of propagation on the line.
After the switch has been moved back to the B position, we now have a rectangular pulse of electrical energy, of voltage amplitude V_i and width Δt, propagating down the line at a velocity v_p, towards the load, in the transmission line system shown in figure 2.



Propagation and reflection on a transmission line

In the general case, when the pulse arrives at the load it will encounter a step change in impedance (an impedance *discontinuity*) at the point of termination, i.e. $R_L \neq Z_o$. Kirchhoff's laws and Ohm's law must always be satisfied at any point in space and time. We therefore have the following boundary conditions at the instant the voltage pulse arrives at the load :

$$V_L = I_L R_L \tag{6}$$

and :

$$V_i = I_i Z_o \tag{7}$$

but, at the instant the pulse arrives at the load, $V_L = V_i$, we therefore have, from (6) and (7) :

$$I_L R_L = I_i Z_o \tag{8}$$

We know, however, that in the general case $R_L \neq Z_o$, so (8) implies that $I_L \neq I_i$. How, then, can Kirchhoff's current law be satisfied at the load? In order to account for the difference between I_L and I_i at the load, we postulate a new term, I_r, which relates to I_L and I_i as follows :

$$I_L = I_i - I_r \tag{9}$$

Propagation and reflection on a transmission line

If we assume that no electromagnetic energy is radiated from the load, the only place for the balancing current, I_r, to flow is back down the line towards the source. We therefore deduce that I_r (and the corresponding voltage, $V_r = Z_o I_r$) constitutes a *reflected* energy pulse that travels back down the line towards the source.

Applying Kirchhoff's voltage law at the load now gives the following :

$$V_L = V_i + V_r \tag{10}$$

Combining (10) with (6), (7) and (9) results in :

$$\frac{V_i + V_r}{R_L} = \frac{V_i}{Z_o} - \frac{V_r}{Z_o} \tag{11}$$

A simple rearrangement of (11) results in :

$$V_i \left[\frac{1}{Z_o} - \frac{1}{R_L} \right] = V_r \left[\frac{1}{Z_o} + \frac{1}{R_L} \right] \tag{12}$$

Propagation and reflection on a transmission line

Of particular interest is the ratio of reflected voltage amplitude to incident voltage amplitude, which can be found from (12) as :

$$\frac{V_r}{V_i} = \frac{R_L - Z_o}{R_L + Z_o} = \Gamma_L \tag{13}$$

This ratio is referred to as the *load voltage reflection coefficient* or simply the *load reflection coefficient*, and is given the symbol Γ_L. From (13), we can observe that if $R_L = Z_o$ then $\Gamma_L = 0$. In other words, if the load is equal to the characteristic impedance of the line then there will be no reflected energy and all the incident energy will be absorbed in the load. Under this condition, we say that the line is *matched*. As far as the source is concerned, a matched line, of any length, is indistinguishable from a line of infinite length.

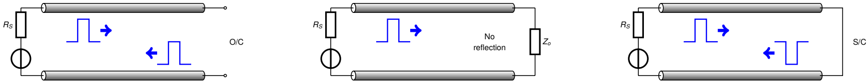
We can further deduce from (13) that if $R_L < Z_o$ then Γ_L will be negative, whereas if $R_L > Z_o$ then Γ_L will be positive. This means that the polarity of the reflected pulse from a load that is less than Z_o will be of the opposite polarity to the incident pulse, whereas the polarity of the reflected pulse from a load that is greater than Z_o will be the same polarity as the incident.

Propagation and reflection on a transmission line

In extremis, the maximum value of load impedance is infinity (i.e. an open circuit) and the minimum value of load impedance is zero (i.e. a short circuit). The reflection characteristics of a line with extreme load values is summarised in table 1 and illustrated in figure 3.

Table 1 : Reflection coefficient extremes

Load	Γ_L	Reflected pulse
Open circuit ($Z_L = \infty$)	1	Same amplitude, same polarity as the incident pulse
Matched ($Z_L = Z_o$)	0	No reflected pulse
Short circuit ($Z_L = 0$)	-1	Same amplitude but opposite polarity to incident pulse



(a) Open circuit ($Z_L = \infty$) (b) Matched ($Z_L = Z_o$) (c) Short circuit ($Z_L = 0$)

Figure 3 : Incident and reflected pulses with extreme load values

Propagation and reflection on a transmission line

We now observe a reflected voltage pulse, of width Δt and voltage amplitude that we will now refer to as "V_{r1}" traveling back down the line from the load towards the source. Employing (13), the voltage amplitude of this reflected pulse is given by :

$$V_{r1} = \Gamma_L V_i \tag{14}$$

When this reflected pulse arrives back at the source, at a time $t = 2\tau$ later, some of the pulse energy will be absorbed in R_S, and the remainder of the energy will be reflected back down the line towards the load again, this time in the form of another voltage pulse of width Δt and voltage amplitude which we will refer to as V_{r2}, as follows:

$$V_{r2} = \Gamma_L \Gamma_S V_i \tag{15}$$

Where Γ_S is the *source voltage reflection coefficient*, which is defined, with respect to Z_o, similarly to (13), as :

$$\Gamma_S = \frac{R_S - Z_o}{R_S + Z_o} \tag{16}$$

This secondary pulse will be reflected again at the load resulting in another reflected voltage pulse, of further reduced amplitude, travelling back towards the source again. This process continues indefinitely as long as R_S and R_L remain connected to the line as in figure 2.

Propagation and reflection on a transmission line

Provided both source and load terminations are purely resistive (i.e. $0 < R_S < \infty$ and $0 < R_L < \infty$), the magnitudes of Γ_S and Γ_L will always be less than or equal to unity, each successive pulse will be of smaller amplitude than the one before, ultimately becoming vanishingly small (but never zero).

If we were to place a suitable voltage measuring instrument exactly half way along a line terminated with $R_S < Z_o$ and $R_L < Z_o$, we would observe the picture shown in figure 4.

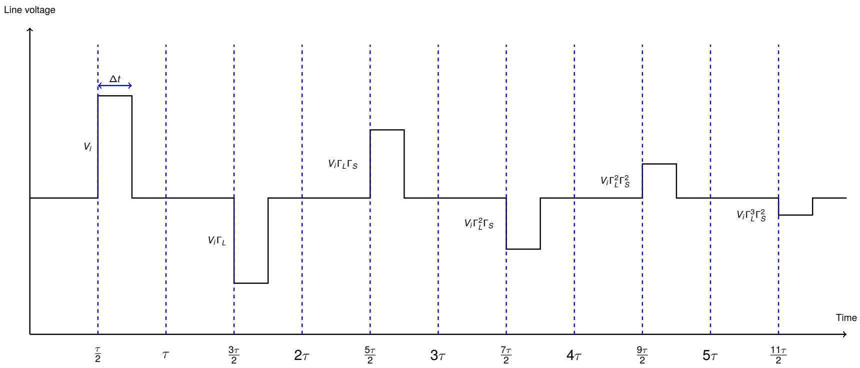


Figure 4 : Voltage pulse and reflections, as seen at a point exactly half way along the transmission line with $R_L < Z_o$ and $R_S < Z_o$

Propagation and reflection on a transmission line

It is instructive to now consider the sequence of events that occurs when we connect the DC source in figure 1 to a transmission line terminated with an open circuit. Let us assume we connect the source to the line at time $t = 0$ and leave it permanently connected. Since there is an open circuit at the end of the line, there are two things we can say for certain :

1. At the exact instant the switch is closed, at $t = 0$, the voltage across the input of the transmission line, V_{in}, will be given by (2), which implies (for finite values of Z_o and R_S) that $V_{in} \neq V_S$.
2. In the steady state condition, i.e. some time after the switch has been closed, the voltage across the input of the transmission line, V_{in}, and indeed at all points along the line, must be equal to V_S.

How does the voltage change as we transition from state (1) to state (2) above? Consider the *transient analysis* shown on the next slide (figure 5)[1].

Propagation and reflection on a transmission line

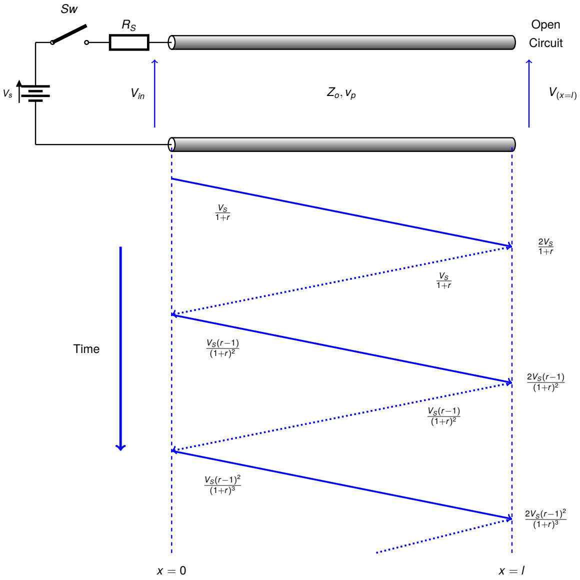


Figure 5 : Open circuit transmission line, DC driven

Propagation and reflection on a transmission line

At the instant the switch is closed ($t = 0$), the voltage across the input of the transmission line is given by :

$$V_{in} = V_S \left(\frac{Z_o}{R_S + Z_o} \right) = V_S \left(\frac{1}{1 + r} \right) \tag{17}$$

Where $r = R_S / Z_o$ or the 'normalised' source resistance[1]. The voltage wave described by (17) propagates along the transmission line in a finite time and arrives at the far end of the line a fixed time, $t = \tau$, later. Since the line is terminated in an open circuit ($\Gamma_L = 1$), this voltage is totally reflected back towards the source as illustrated in figure 5.

When the reflected voltage propagating back down the line reaches the source, it sees a reflection coefficient, Γ_S, given by :

$$\Gamma_S = \frac{R_S - Z_o}{R_S + Z_o} = \frac{r - 1}{r + 1} \tag{18}$$

Hence, a fraction of this voltage is reflected back down the line towards the open circuit with a value given by :

$$\left(\frac{V_S}{1 + r} \right) \Gamma_S = \frac{V_S}{1 + r} \left(\frac{r - 1}{r + 1} \right) = V_S \frac{r - 1}{(r + 1)^2} \tag{19}$$

which in turn gets reflected at the open circuit and so on, as shown in figure 5.

Propagation and reflection on a transmission line

The total voltage at $x = l$ builds up as t increases. The final voltage, at $t = \infty$, is given by the addition of all these partial voltages, that is to say :

$$V_{(x=l)} = 2V_S \left(\frac{1}{r+1} + \frac{r-1}{(r+1)^2} + \frac{(r-1)^2}{(r+1)^3} + \dots \right) \tag{20}$$

or :

$$V_{(x=l)} = \left(\frac{2V_S}{r+1} \right) \sum_{k=0}^{\infty} \left(\frac{r-1}{r+1} \right)^k \tag{21}$$

Equation (21) contains a sum of an infinite geometric series, which can be replaced by its equivalent, resulting in :

$$V_{(x=l)} = \left(\frac{2V_S}{r+1} \right) \left(\frac{r+1}{2} \right) \tag{22}$$

$$= V_S \tag{23}$$

In other words, the voltage at the open end of the transmission line in figure 5 will eventually come to equal the source voltage, V_S, although this will not happen instantaneously. What is interesting is how the output voltage gets to V_S and what happens in the first few moments after the switch is closed, which depends on the value of r.

Propagation and reflection on a transmission line

- Figure 6, illustrates the voltage at the load end of the transmission line ($x = l$) versus time for three values of the ratio r, namely $r = 4$, $r = 1$ and $r = 0.25$[1].
- We observe that in all three cases the voltage at the load tends to V_S, as we would expect. Note that in figure 6(b) the voltage at the load equals V_S for any $t > \tau$, since $r = 1$ means that $R_S = Z_o$.
- When the first reflection arrives back at the source, it is completely absorbed by the source resistance and there are no subsequent reflections.

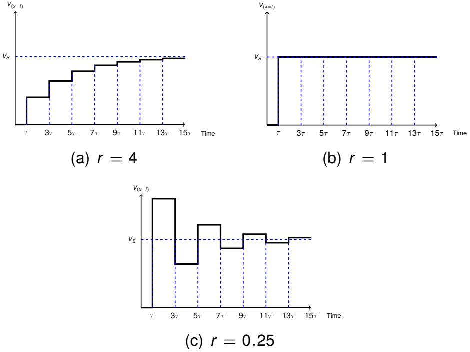
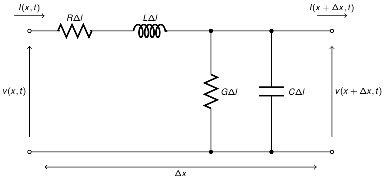


Figure 6 : Load voltage development on an open circuit terminated line for various values of normalised source resistance, r

Transmission Line Analysis

Transmission line analysis involves modelling the transmission line as an infinite series of two-port elementary components, each representing an infinitesimally short segment of length Δx :



- The distributed resistance, R. of the conductors is represented by a series resistor (expressed in ohms per unit length).
- The distributed inductance, L, (due to the magnetic field around and inside the conductors) is represented by a series inductor (henries per unit length).
- The distributed capacitance, C, between the two conductors is represented by a shunt capacitor (farads per unit length).
- The distributed conductance, G, of the dielectric material separating the two conductors is represented by a shunt resistor between the signal wire and the return wire (siemens per unit length).

Table of Contents

The Telegrapher's Equations

The line voltage V(x) and the current I(x) in the previous slide can be expressed in the frequency domain using the Telegrapher's Equations :

$$\frac{\partial V(x)}{\partial x} = -(R + j\omega L)I(x) \tag{24}$$

$$\frac{\partial I(x)}{\partial x} = -(G + j\omega C)V(x) \tag{25}$$

- I(x) and V(x) describe the current and voltage along the transmission line, as a function as position x.
- The functions I(x) and V(x) are complex, where the magnitude and phase of the complex functions describe the magnitude and phase of the sinusoidal time function ejÏĽt.
- Only those functions of I(x) and V(x) that satisfy the telegraphers equations can exist on a transmission line.

Solution of the Telegrapher's Equations

What functions I (x) and V (x) satisfy both telegrapher's equations? We will first combine the telegrapher equations to form one differential equation for V (x) and another for I(x). First, take the derivative with respect to x of the first telegrapher equation:

$$\frac{\partial}{\partial x}\left\{\frac{\partial V(x)}{\partial x} = -(R + j\omega L)I(x)\right\} \tag{26}$$

$$\frac{\partial^2 V(x)}{\partial x^2} = -(R + j\omega L)\frac{\partial I(x)}{\partial x} \tag{27}$$

Solution of the Telegrapher's Equations

We the substitute $\frac{\partial I(x)}{\partial x}$ from the second telegrapher's equation into (27), on the previous slide, to give the following equation in terms of V(x) only :

$$\frac{\partial^2 V(x)}{\partial x^2} = (R + j\omega L)(G + j\omega C)V(x) \tag{28}$$

$$\frac{\partial^2 V(x)}{\partial x^2} = \gamma^2 V(x) \tag{29}$$

Where γ is the Propagation Constant defined by :

$$\boxed{\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} = \alpha + j\beta} \tag{30}$$

Where :
α is called the *attenuation* (or loss) factor
β is called the *phase* (or velocity) factor

Solution of the Telegrapher's Equations

We can now restate both telegrapher's equations as follows :

$$\frac{\partial^2 V(x)}{\partial x^2} - \gamma^2 V(x) = 0 \tag{31}$$

$$\frac{\partial^2 I(x)}{\partial x^2} - \gamma^2 I(x) = 0 \tag{32}$$

These are in the form of wave equations and are referred to as the transmission line wave equations. We need to find functions V(x) and I(x) which satisfy the above. Consider the first equation above : two possible solutions are the functions

$$V(x) = e^{-\gamma x} \tag{33}$$

and

$$V(x) = e^{+\gamma x} \tag{34}$$

Which represent sinusoidal waves travelling in the negative x direction and the positive x direction respectively.

Solution of the Telegrapher's Equations

Since the transmission line wave equation is a linear differential equation, a weighted superposition of the two solutions is also a solution. So the general solution to these wave equations is:

$$V(x) = V_o^+ e^{+\gamma x} + V_o^- e^{+\gamma x} \tag{35}$$

$$I(x) = I_o^+ e^{+\gamma x} + I_o^- e^{+\gamma x} \tag{36}$$

The two terms in each solution describe two waves propagating in opposite directions on the transmission line. By convention the first term represents a wave propagating in the positive (+x) direction. The second term represents a wave propagating in the opposite direction (-x). We can rewrite the above as :

$$V(x) = V^+(x) + V^-(x) \tag{37}$$

$$I(x) = I^+(x) + I^-(x) \tag{38}$$

Table of Contents

Lossless Transmission Line

Recall equation (30)

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} = \alpha + j\beta \quad (39)$$

If the line is lossless then, by definition, R=0 and G=0. The propagation constant then simplifies to :

$$\gamma = j\omega\sqrt{LC} \quad (40)$$

From equation (28) and equation (42) the second order steady state Telegrapher's Equations for a lossless line therefore become :

$$\frac{\partial^2 V(x)}{\partial x^2} - \omega^2 LCV(x) = 0 \quad (41)$$

$$\frac{\partial^2 I(x)}{\partial x^2} - \omega^2 LCI(x) = 0 \quad (42)$$

Table of Contents

Derivation of the Characteristic Impedance

- We introduced the concept of *characteristic Impedance* earlier in this chapter, as the 'mystery' impedance seen by the source when looking into an infinitely long transmission line.
- We will now proceed to derive an equation for the characteristic impedance of a line in terms of the primary line parameters introduced in the previous section.
- Consider a sinusoidal voltage source connected to an infinitely long transmission line which can be modelled as an infinite series of the unit elements shown in figure **??**.
- In figure 7 we have separated out the first of these infinitesimal elements to aid the analysis to follow.

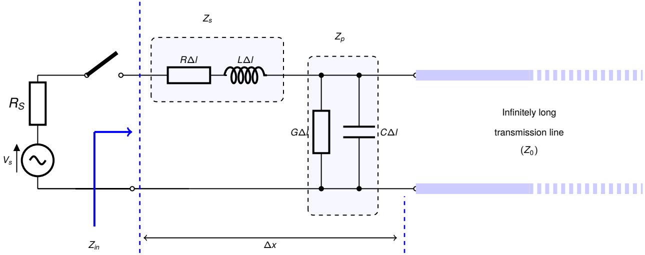


Figure 7 : Characteristic impedance of an infinitely long transmission line

Characteristic Impedance

- Consider the instant, t=0, when the switch in figure 7 is closed.
- In order for current to flow into the line, the source must see a finite impedance.
- We define this impedance, i.e. the impedance of an infinitely long line as the **Characteristic Impedance**, which is denoted by Z_o.
- Considering the circuit on the previous slide we can therefore define the impedance seen by the source at the instant t=0 as :

$$Z_{in} = Z_s + \frac{Z_p Z_o}{Z_p + Z_o} \tag{43}$$

Where Z_s and Z_p are the unit series and parallel elements defined by :

$$Z_p = \frac{1}{(G + j\omega C)\Delta x} \tag{44}$$

$$Z_s = (R + j\omega L)\Delta x \tag{45}$$

Characteristic Impedance

For an infinitely long line, $Z_{in} = Z_o$ therefore :

$$Z_o = Z_s + \frac{Z_p Z_o}{Z_p + Z_o} \tag{46}$$

Thus :

$$Z_o^2 - Z_o Z_s - Z_p Z_s = 0 \tag{47}$$

$$Z_o = \frac{Z_s \pm \sqrt{Z_s^2 + 4 Z_p Z_s}}{2} \tag{48}$$

$$Z_o = \frac{(R + j\omega L)\Delta x}{2} \pm \frac{1}{2}\sqrt{[(R + j\omega L)\Delta x]^2 + 4\frac{(R + j\omega L)}{(G + j\omega C)}} \tag{49}$$

as $\Delta x \to 0$ equation (49) reduces to :

$$\boxed{Z_o = \sqrt{\frac{(R + j\omega L)}{(G + j\omega C)}}} \tag{50}$$

Table of Contents

Terminated Transmission Line

- We will now further investigate the behaviour of a transmission line, of characteristic impedance Z_o, terminated with an arbitrary load, Z_L, under conditions of steady state sinusoidal excitation, as shown in figure 8.
- We will firstly deal with the general case of a lossy line, having a complex propagation constant $\gamma = \alpha + j\beta$. The specific case of a lossless line will be covered in the next section.

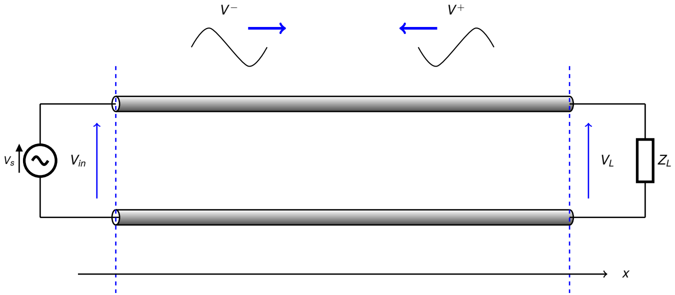


Figure 8 : Transmission line with complex load

Terminated Transmission Line

Under these steady state conditions we will observe a standing wave of voltage and current on the line, i.e. voltage and current are functions of position, x, only. The expressions for voltage and current on the line are given by (**??**) and (**??**) :

$$V(x) = V^{+}e^{-\gamma x} + V^{-}e^{\gamma x} \quad (\textbf{??})$$

$$I(x) = I^{+}e^{-\gamma x} - I^{-}e^{\gamma x} \quad (\textbf{??})$$

Where V^{+} and V^{-} are the amplitudes of the forward and reverse propagating voltage waves on the line, and I^{+} and I^{-} are the amplitudes of the corresponding current waves. At the load, i.e. at $x = 0$, we have :

$$V_L = V^{+} + V^{-} \quad (51)$$

$$I_L = I^{+} - I^{-} \quad (52)$$

But $V^{+} = Z_o I^{+}$ and $V^{-} = Z_o I^{-}$, hence (52) can be written as :

$$I_L = \frac{1}{Z_o}(V^{+} - V^{-}) \quad (53)$$

Terminated Transmission Line

Combining (51) and (53) we have, at the load :

$$\frac{V_L}{I_L} = Z_o \frac{(V^+ + V^-)}{(V^+ - V^-)} \tag{54}$$

At this point, we note that $V_L/I_L = Z_L$ and $V^-/V^+ = \Gamma_L$, the voltage reflection coefficient at the load. We can therefore rewrite (54) as :

$$Z_L = Z_o \frac{(1 + \Gamma_L)}{(1 - \Gamma_L)} \tag{55}$$

Which, when re-arranged, is consistent with (13), i.e. :

$$\Gamma_L = \frac{(Z_L - Z_o)}{(Z_L + Z_o)} \tag{56}$$

This gives us the voltage reflection coefficient at only one location on the line, namely at the load where we have defined $x = 0$. What we now wish to do is generalise this to obtain an expression for the reflection coefficient at any point on the line.

Terminated Transmission Line

We can write the voltage reflection coefficient at any arbitrary distance, $x = -l$, back from the load, towards the source, as the ratio of reverse and forward propagating voltage waves, i.e. :

$$\Gamma(l) = \frac{V^- e^{-\gamma l}}{V^+ e^{\gamma l}} = \frac{V^-}{V^+} e^{-2\gamma l} \tag{57}$$

We have already established that $V^-/V^+ = \Gamma_L$, the reflection coefficient of the load, so we can write :

$$\Gamma(l) = \Gamma_L e^{-2\gamma l} \tag{58}$$

Which is the reflection coefficient looking into the line at a distance, l, from the load.

Input impedance at an arbitrary point on a line

Let us define $Z_{in}(l)$ as the impedance looking into a terminated line, towards the load, at location $x = -l$. We will now refer to the normalised input impedance at this point which we shall denote by $z_{in}(l)$, and define as follows :

$$z_{in}(l) = \frac{Z_{in}(l)}{Z_o} = \frac{V(l)}{I(l)Z_o} \tag{59}$$

Applying (**??**) and (**??**) to (59) gives :

$$z_{in}(l) = \frac{V^+e^{\gamma l} + V^-e^{-\gamma l}}{V^+e^{\gamma l} - V^-e^{-\gamma l}} \tag{60}$$

Dividing through by V^+ and applying (57),results in :

$$z_{in}(l) = \frac{e^{\gamma l} + \Gamma_L e^{-\gamma l}}{e^{\gamma l} - \Gamma_L e^{-\gamma l}} \tag{61}$$

Which we can simplify as :

$$z_{in}(l) = \frac{1 + \Gamma_L e^{-2\gamma l}}{1 - \Gamma_L e^{-2\gamma l}} \tag{62}$$

Input impedance at an arbitrary point on a line

Replacing Γ_L with (55) and noting that $e^x = cosh(x) + \sinh(x)$, we can rewrite (62) as :

$$z_{in}(l) = \frac{Z_L \cosh(\gamma l) + Z_o \sinh(\gamma l)}{Z_o cosh(\gamma l) + Z_L \sinh(\gamma l)} \quad (63)$$

Which can be further rewritten as :

$$z_{in}(l) = \frac{Z_L + Z_o \tanh(\gamma l)}{Z_o + Z_L \tanh(\gamma l)} \quad (64)$$

In the special case of a lossless line, where $\alpha = 0$ and thus $\gamma = j\beta$, (64) reduces to :

$$\boxed{z_{in}(l) = \frac{Z_L + jZ_o \tan(\beta l)}{Z_o + jZ_L \tan(\beta l)}} \quad (65)$$

Special cases : Short Circuit Transmission Line

When the line is terminated with a short circuit we have $Z_L = 0$. Therefore (65) simply reduces to :

$$Z_{in} = jZ_o \tan \beta l \qquad (66)$$

The normalised input impedance vs electrical length for an open circuit transmission line is shown in figure 9.

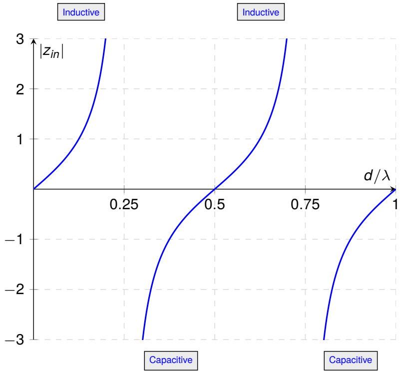


Figure 9 : Normalised input impedance vs electrical length for a short circuit transmission line

Special cases : Short Circuit Transmission Line

- Figure 10 shows an enlarged portion of figure 9 around the origin.
- We can see that, if the electrical length of the line is short enough (say, less than 0.1λ), then (66) is an approximately linear function of β, and thus Z_{in} is approximately proportional to frequency.
- This means that such short sections of short circuit terminated line can be used to substitute for an inductor.
- This is a common technique in the design of Monolithic Microwave Integrated Circuits (MMIC), as a short section of transmission line takes up less space on a semiconductor substrate than, say, a spiral inductor.

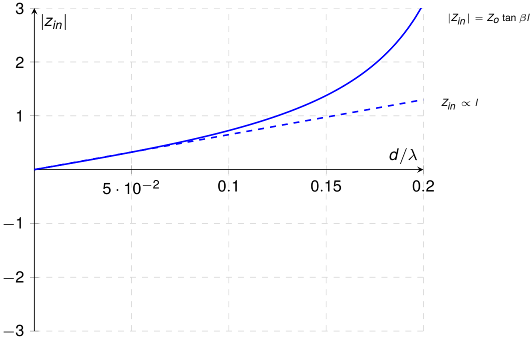


Figure 10 : Use of a short section of short circuit terminated transmission line as an inductor

Special cases : Open Circuit Transmission Line

When the line is terminated with an open circuit we have $Z_L = \infty$. Therefore (65) becomes :

$$Z_{in} = \lim_{Z_L \to \infty} \left(Z_o \frac{1 + j\dfrac{Z_o}{Z_L} \tan \beta l}{\dfrac{Z_o}{Z_L} + j \tan \beta l} \right) \tag{67}$$

$$Z_{in} = \frac{Z_o}{j \tan \beta l} \tag{68}$$

$$Z_{in} = -jZ_o \cot \beta l \tag{69}$$

The normalised input impedance vs electrical length for a open circuit transmission line is shown in figure 11.

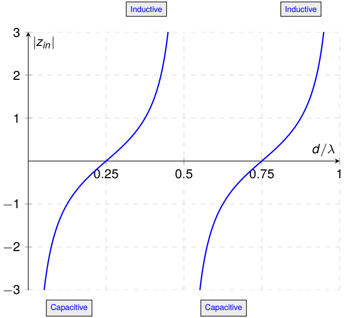


Figure 11 : Normalised input impedance vs electrical length for an open circuit transmission line

Special cases : Matched Transmission Line

When the line is terminated with a load equal to the characteristic impedance ($Z_L = Z_o$). The input impedance of such a line, from (65), is simply :

$$Z_{in} = Z_o \tag{70}$$

In other words, a line terminated in the characteristic impedance will appear to the source as if it were an infinitely long line of characteristic impedance Z_o.

Example Calculation

A load of 100Ω is connected to a lossless transmission line with a characteristic impedance of 50Ω and length of $0.5l/\lambda$. Calculate :

(a) The reflection coefficient at the load

(b) VSWR

(c) The input impedance of the line

Solution :

(a)

$$\Gamma_L = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{100 - 50}{100 + 50} = \frac{1}{3} \tag{71}$$

(b)

$$VSWR = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|} = 2 \tag{72}$$

(c)

$$\Gamma_L = Z_o \frac{Z_L + Z_o tan(\beta l)}{Z_o + Z_L tan(\beta l)} \qquad where = \beta l = \frac{2\pi}{\lambda_g} = \pi$$

$$Therefore : tan(\beta l) = 0 \qquad therefore : Z_{in} = Z_L = 100\Omega$$

Table of Contents

Characteristic impedance of lossy lines

- A lossy line is one that contains appreciable series resistance and/or shunt conductance. Different frequencies travel at different speeds on lossy lines, hence, any signal containing multiple frequency components, such as a rectangular pulse, will not remain rectangular but will become distorted in shape as it travels down the line.
- The presence of losses changes the velocity of propagation and causes the wave to be attenuated as it travels. Therefore, any lossy line of sufficient length, will appear to be well matched to a source of the same characteristic impedance, irrespective of the actual load impedance.
- We can extend the analysis covered in the previous sections by re-introducing the attenuation constant, α. The propagation constant is now given by equation **??**. The reflection coefficient looking into the line at a distance, x, back from the load Γ_L is given by (58).

In the case of a lossless line, $\alpha = 0$ and so (58) reduces to:

$$\Gamma(x) = \Gamma_L e^{-2\beta x} \tag{73}$$

In the case of a lossy line, we cannot ignore α, so (58) should be written as:

$$\begin{aligned}\Gamma(x) &= \Gamma_L e^{-2(\alpha + j\beta)x} \\ &= \Gamma_L e^{-2\alpha x} e^{-j2\beta x}\end{aligned} \tag{74}$$

Dispersion

- Comparing (74) and (73) we can see that the introduction of the attenuation constant, α, has the effect of reducing the magnitude of $\Gamma(x)$ exponentially as x increases. In the limit as $x \to \infty$ we find that the magnitude of the reflection coefficient looking into the line tends to zero. In other words, any sufficiently long length of lossy line will appear to be matched, irrespective of the value of the load.
- This makes sense when we consider that, in a lossy line, energy in both the forward and reflected waves is being continuously dissipated as they travel along the line. Any reflections from an arbitrary load will be reduced exponentially with physical length, thereby making the load appear better matched than it would be if the line was lossless.
- Another implication of line loss can be understood by considering equation (**??**). If either $R \neq 0$ or $G \neq 0$ then the characteristic impedance will be complex. One implication of this is that a purely resistive source can never be perfectly matched to a lossy line, irrespective of the electrical length.

Dispersion

- In general, the phase velocity, v_p, is a function of frequency. In other words, in the case of a signal containing many frequency components, which is any signal other than a single frequency sinusoid, the various frequency components will travel down the line at different speeds. The result of this is that the signal will become 'dispersed', i.e. the various components of the signal will arrive at the load at different times.
- This can be catastrophic for signals that are composed of many frequency components, such as square pulses (i.e. digital signals) which end up being 'smeared out' in time as they travel down the line. If the line is long enough, square pulses will end up being so distorted that they will be unrecoverable at the receiver.
- This implies that the only way to prevent dispersion is to use a lossless line, so that v_p is independent of frequency. This suggest that dispersion is unavoidable, since all real-world lines have losses.

Dispersion

In the course of his work on the transatlantic cable problem outlined in the introduction to this chapter, Oliver Heaviside showed that a transmission line would be dispersionless if the line parameters exhibited the following ratio[3]:

$$\frac{R}{L} = \frac{G}{C} \tag{75}$$

This is called the *Heaviside Condition*, which can be verified by substituting (75) into (**??**) :

$$\begin{aligned}
\gamma &= \sqrt{LC(R/L + j\omega)(G/C + j\omega)} \\
&= \sqrt{LC(R/L + j\omega)(R/L + j\omega)} \\
&= (R/L + j\omega)\sqrt{LC} \\
&= \boxed{R\sqrt{\frac{C}{L}} + j\omega\sqrt{LC}}
\end{aligned} \tag{76}$$

Dispersion

Given that $\gamma = \alpha + j\beta$ we can write (76) as :

$$\alpha = R\sqrt{\frac{C}{L}} \tag{77}$$

and :

$$\beta = \omega\sqrt{LC} \tag{78}$$

We note that (78) is the same as (**??**), meaning that we have achieved the same result as for a lossless line. We also have the following phase velocity :

$$\upsilon_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}} \tag{79}$$

Since the phase velocity is independent of frequency, we have created a lossy line that behaves like a lossless line, provided that (75) is satisfied. Thus Oliver Heaviside showed that we can eliminate dispersion from a lossy transmission line by adding series inductors periodically along the line so as to satisfy (75).

Table of Contents

Power Considerations

If a lossless line is matched, i.e. $Z_L = Z_o$, then all of the power contained in the incident wave will be delivered to the load, i.e. no power will be reflected back from the load. If the line is not matched, i.e. $Z_L \neq Z_o$, then some of the power incident at the load will be reflected back down the line towards the source. In general, the power dissipated in a load is given by :

$$P_L = \frac{1}{2} Re(V_L I_L^*) \tag{80}$$

Where V_L and I_L represent complex load voltage and current respectively and $(^*)$ represents the complex conjugate. (80) can be expressed in terms of forward and reflected travelling waves on the line arriving at the load as follows :

$$P_L = \frac{1}{2} Re[(V^+ + V^-)(I^+ - I^-)^*] \tag{81}$$

$$P_L = \frac{1}{2} Re\left[(V^+ + V^-)\left(\frac{(V^+ - V^-)^*}{Z_o}\right)\right] \tag{82}$$

$$P_L = \frac{1}{2} Re\left[\frac{|V^+|^2}{Z_o}(1 + \Gamma_L)(1 - \Gamma_L)^*\right] \tag{83}$$

Which simplifies to :

$$P_L = \frac{1}{2} \frac{|V^+|^2}{Z_o}(1 - |\Gamma_L|^2) \tag{84}$$

Power Considerations

The physical interpretation of (84) is that some portion of the incident power will be absorbed in the load and the remainder will be reflected back towards the source. (84) will apply, in general, to any discontinuity at any point in a transmission line. In such situations, some portion of the incident power will be reflected back towards the source, and some will be transmitted onwards down the line. The respective powers will be determined by the magnitude of the reflection coefficient at the discontinuity, Γ_x, as illustrated in figure 12.

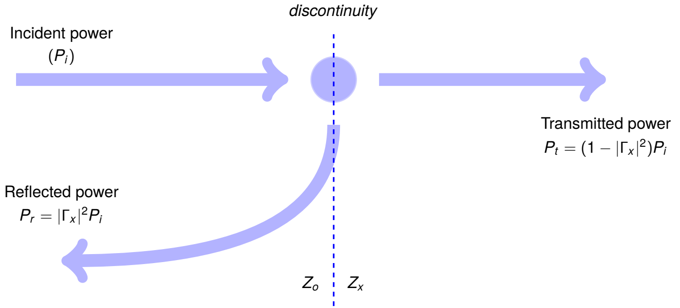


Figure 12 : Generalised picture of power reflection and transmission at a transmission line discontinuity

Where the reflection coefficient at the discontinuity is defined by :

$$Z_x - Z_o$$

Power Considerations

When considering power flows, another parameter that is often used to describe the degree of match or mismatch on a transmission line is *return loss*, which is basically the ratio of incident to reflected power, expressed in dB :

$$RL_{dB} = 10\log_{10}\left[\frac{P_i}{P_r}\right] \tag{86}$$

In terms of the reflection coefficient, Γ, at the load or discontinuity in question, we have:

$$RL_{dB} = 10\log_{10}\left[\frac{1}{|\Gamma|^2}\right] = -20\log_{10}|\Gamma| \tag{87}$$

The higher the return loss the better the match, which can be interpreted as more of the incident power being "lost" in the load, so less power being reflected back towards the source.

It is useful to now extend table 1 to include VSWR and return loss as shown in table 2 :

Table 2 : Reflection coefficient, VSWR and return loss extremes

Load type	Z_L	Γ_L	VSWR	Return loss
Open circuit	∞	1	∞	0dB
Matched	Z_o	0	1	∞
Short circuit	0	-1	∞	0dB

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