Lecture 7 - Gain and stability of active networks

Microwave Active Circuit Analysis and Design

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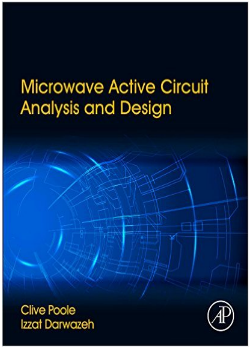
Academic Press Inc.



Intended Learning Outcomes

- Knowledge
  - Understand the definitions of conditional and unconditional stability and be able to determine the stability of a microwave active device using defined stability criteria.
  - Understand and be able to calculate the various different definitions of power gain in small signal amplifiers.
  - Understand that an active two-port network has the possibility of becoming unstable, and therefore requires different treatment than a passive two-port.
  - Understand the various definitions of power at the input and output ports, and the corresponding definitions of 'power gain' using either immittance parameters or *S*-parameters.
- Skills
  - Be able to calculate the transducer gain, available gain, operating power gain and maximum available power gain of any two-port network with given source and load terminations based on immittance parameters or *S*-parameters.
  - Be able to calculate the optimum terminations to achieve simultaneous conjugate matching, and therefore maximum available gain, from an unconditionally stable transistor.
  - Be able to determine whether any two-port network is unconditionally stable or potentially unstable given a set of two-port immittance parameters or *S*-parameters.

Table of Contents

Voltage gain of an active two-port

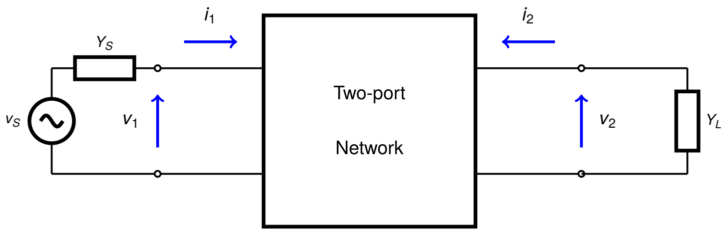


Figure 1 : Two-port power gain definitions

The voltage gain of the two-port in figure 1 is given by :

$$A_v = \frac{v_2}{v_1} \quad (1)$$

From figure 1 we can write :

$$i_2 = -v_2 Y_L \quad (2)$$

But i_2 is also related to v_1 and v_2 via the Y-parameters of the two-port, so we can also write:

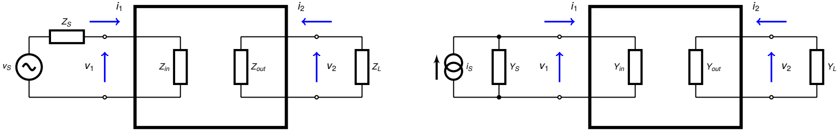
$$i_2 = Y_{21} v_1 + v_2 Y_{22} \quad (3)$$

Solving (2) and (3) as simultaneous equations and applying (1) gives the small signal voltage gain for the two-port as follows:

$$A_v = \frac{v_2}{v_1} = \frac{-Y_{21}}{Y_L + Y_{22}} \quad (4)$$

Table of Contents

Power absorbed by the amplifier input (P_{in})



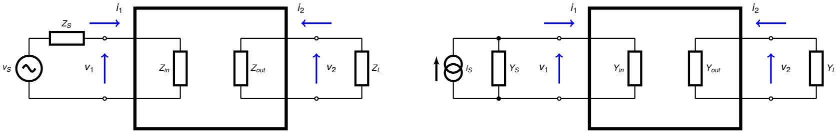
(a) Z-parameters (b) Y-parameters

Figure 2 : Two-port power gain definitions

P_{in} is the power actually delivered to the input port of the amplifier, irrespective of whether the source is conjugately matched to the input port.

Z-parameters	Y-parameters
$P_{in} = \|i_1\|^2 Re(Z_{in}) \quad (5)$	$P_{in} = \|v_1\|^2 Re(Y_{in}) \quad (6)$
$P_{in} = \|v_S\|^2 \frac{Re(Z_{in})}{\|Z_S + Z_{in}\|^2} \quad (7)$	$P_{in} = \|i_S\|^2 \cdot \frac{Re(Y_{in})}{\|Y_S + Y_{in}\|^2} \quad (8)$

Power available from the generator (P_{AVS})



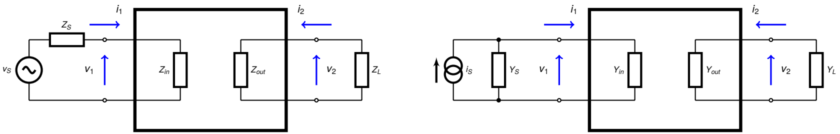
(a) Z-parameters (b) Y-parameters

Figure 3 : Two-port power gain definitions

P_{AVS} is the maximum power available from the source when it is conjugately matched to the input impedance of the amplifier. By setting $Z_S = Z_{in}^*$ in (7) or $Y_S = Y_{in}^*$ in (8) we get:

Z-parameters	Y-parameters
$$P_{AVS} = \frac{\lvert v_S \rvert^2}{4Re(Z_S)} \quad (9)$$	$$P_{AVS} = \frac{\lvert i_S \rvert^2}{4Re(Y_S)} \quad (10)$$

Power absorbed by the load (P_L)



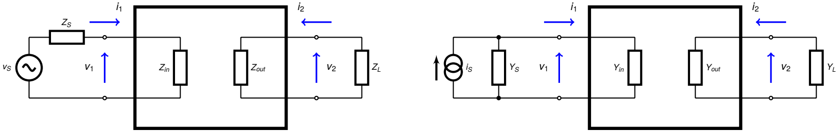
(a) Z-parameters (b) Y-parameters

Figure 4 : Two-port power gain definitions

P_L is the power actually delivered to the load, irrespective of whether the load is conjugately matched to the output of the two-port. This is defined as:

Z-parameters	Y-parameters
$P_L = \|i_2\|^2 \cdot Re(Z_L)$ (11)	$P_L = \|v_2\|^2 \cdot Re(Y_L)$ (12)
$P_L = \|v_2\|^2 \cdot \frac{Re(Z_L)}{\|Z_{out} + Z_L\|^2}$ (13)	$P_L = \|i_2\|^2 \cdot \frac{Re(Y_L)}{\|Y_{out} + Y_L\|^2}$ (14)

Power available from the amplifier output (P_{AVN})



(a) Z-parameters (b) Y-parameters

P_{AVN} is the maximum power available from the amplifier output when it is conjugately matched to the load. By setting $Z_S = Z_{in}^*$ in (9) or $Y_S = Y_{in}^*$ in (10) we get :

Z-parameters	Y-parameters
$$P_{AVN} = \frac{\lvert v_2 \rvert^2}{4Re(Z_{out})} \qquad (15)$$	$$P_{AVN} = \frac{\lvert i_2 \rvert^2}{4Re(Y_{out})} \qquad (16)$$

Power gain definitions

Using the power definitions (5) to (16), we can state the four most common definitions of two-port power gain as follows[9]:

1. Transducer Power Gain :
$$G_T = \frac{P_L}{P_{AVS}} \tag{17}$$
2. Available Power Gain :
$$G_A = \frac{P_{AVN}}{P_{AVS}} \tag{18}$$
3. Operating Power Gain :
$$G_o = \frac{P_L}{P_{in}} \tag{19}$$

Transducer power gain in terms of Y-parameters

The transducer gain, G_T, refers to the general case of any arbitrary source and load termination. The expression for transducer gain must therefore contain both source admittance, Y_S, and load admittance, Y_L. Applying (10) and (12) to (17) we can write the transducer gain as follows :

$$G_T = \frac{P_L}{P_{AVS}} = 4 \cdot \frac{|v_2|^2}{|i_s|^2} \cdot Re(Y_S)Re(Y_L) \tag{20}$$

We now need to replace the term $|v_2|^2/|i_s|^2$ in (20) by a term involving only the Y-parameters. With reference to figure 5(b) we can write the following :

$$v_1 = \frac{i_s}{Y_S + Y_{in}} \tag{21}$$

By replacing Y_{in} in (21) with (**??**) we can write :

$$v_1 = \frac{i_s}{Y_S + \left[Y_{11} - \frac{Y_{12}Y_{21}}{Y_L + Y_{22}}\right]} \tag{22}$$

$$= \frac{i_s(Y_L + Y_{22})}{(Y_S + Y_{11})(Y_L + Y_{22}) - Y_{12}Y_{21}} \tag{23}$$

Transducer power gain in terms of Y-parameters

We note that v_2 is related to v_1 and i_2 by the definition of the Y-parameters in section **??**, i.e. :

$$i_2 = Y_{22}v_2 + Y_{21}v_1 \tag{24}$$

Rearranging (24) gives:

$$v_2 = \frac{i_2 - Y_{21}v_1}{Y_{22}} \tag{25}$$

From figure 5(b) we can see that i_2 is also equal to $-v_2Y_L$, so we can write :

$$v_2 = \frac{-v_2Y_L - Y_{21}v_1}{Y_{22}} \tag{26}$$

$$= \frac{-Y_{21}v_1}{Y_{22} + Y_L} \tag{27}$$

Substituting (22) into (26) gives :

$$\frac{v_2}{i_s} = \frac{-Y_{21}}{(Y_S + Y_{11})(Y_L + Y_{22}) - Y_{12}Y_{21}} \tag{28}$$

Finally, we substitute (28) back into (20) to get the required expression for transducer gain in terms of Y-parameters as follows :

$$\boxed{G_T = \frac{4|Y_{21}|^2 Re(Y_L) Re(Y_S)}{|Y_{11} + Y_S|^2 |Y_{22} + Y_L|^2}} \tag{29}$$

Transducer power gain in terms of Y-parameters

We can rearrange (29) as follows :

$$G_T = \frac{2Re(Y_S)}{|Y_{11} + Y_S|^2} \cdot |Y_{21}|^2 \cdot \frac{2Re(Y_L)}{|Y_{22} + Y_L|^2} \qquad (30)$$

We can thereby see that the transducer gain is comprised of three factors :

i) an intrinsic gain component : $|Y_{21}|^2$

ii) a source mismatch factor : $\frac{2Re(Y_S)}{|Y_{11} + Y_S|^2}$

iii) a load mismatch factor : $\frac{2Re(Y_L)}{|Y_{22} + Y_L|^2}$

The first is the intrinsic gain of the Two-port when terminated with the reference impedance that is used to measure the Y-parameters (i.e. a short circuit).
The second factor accounts for the degree of mismatch at the input ports. This can be demonstrated by considering what happens as $Y_S \to \infty$, in which case this factor tends to unity. A similar argument applies to the load mismatch factor.

Available power gain in terms of Y-parameters

We derive available power gain, G_A, in terms of Y-parameters by applying (10) and (16) to (18) to obtain :

$$G_A = \frac{P_{AVN}}{P_{AVS}} = \frac{|i_2|^2}{|i_S|^2} \cdot \frac{Re(Y_S)}{Re(Y_{out})} \tag{31}$$

We can see from figure 5(b) that $i_S = -v_1(Y_S + Y_{11})$ (note that we use Y_{11}, not Y_{in} because i_2 is defined here as the short circuit output current, i.e. $Y_L = \infty$ and so $Y_{in} = Y_{22}$). We can therefore write :

$$\frac{i_2}{i_S} = \frac{i_2}{-v_1(Y_S + Y_{11})} = \frac{-Y_{21}}{Y_S + Y_{11}} \tag{32}$$

If we now replace i_2/i_S in (31) with (32) we have :

$$\boxed{G_A = \frac{|Y_{21}|^2}{|Y_S + Y_{11}|^2} \cdot \frac{Re(Y_S)}{Re(Y_{out})}} \tag{33}$$

Operating power gain in terms of Y-parameters

We now apply a similar reasoning to obtain the operating power gain, G_o, in terms of Y-parameters. By applying (6) and (12) to (19) we have :

$$G_o = \frac{P_L}{P_{in}} = \frac{|v_2|^2}{|v_1|^2} \cdot \frac{Re(Y_L)}{Re(Y_{in})} \tag{34}$$

From figure 5(b) we can see that $i_2 = -v_2(Y_L + Y_{22})$. Again, i_1 is defined as the short circuit input current so $Y_S = \infty$ and $Y_{out} = Y_{11}$. We can therefore write :

$$\frac{v_2}{v_1} = \frac{i_2}{-v_1(Y_L + Y_{22})} = \frac{-Y_{21}}{Y_L + Y_{22}} \tag{35}$$

We now replace v_2/v_1 in (34) with (35) to give :

$$\boxed{G_o = \frac{|Y_{21}|^2}{|Y_L + Y_{22}|^2} \cdot \frac{Re(Y_L)}{Re(Y_{in})}} \tag{36}$$

Operating power gain in terms of Y-parameters

Since the input power P_{in} will always be less than or equal to the available source power P_{AVS}, and the available output power P_{AVN} is always greater than or equal to the power actually delivered to the load P_L, we can state the following inequalities relating the three gains :

$$G_T \leq G_o \tag{37}$$

and

$$G_T \leq G_A \tag{38}$$

In other words, for a given set of source and load terminations, the transducer gain will always be the smallest of the three gain figures.
It follows from (37) and (38) that when we maximise G_T , by suitable choice of Y_S and Y_L, then both G_o and G_A will also be maximised. In other words :

$$G_{T_{max}} = G_{o_{max}} = G_{A_{max}} = G_{max} \tag{39}$$

Where G_{max} is the unique value of *maximum available gain* that is a single figure of merit for a given two-port under conditions of simultaneous conjugate matching at both ports.

Simultaneous conjugate matching (Y-parameters)

Assuming we are able to simultaneously optimise the matching at input and output ports, G_{max} is given in terms of Y-parameters by :

$$G_{max} = \frac{|Y_{21}|^2}{2G_{11}G_{22}(1+M) - Re(Y_{12}Y_{12})} \tag{40}$$

Where M is given by:

$$M = \sqrt{1 - \frac{Re(Y_{12}Y_{12})}{G_{11}G_{22}} - \left[\frac{Im(Y_{12}Y_{12})}{2G_{11}G_{22}}\right]^2} \tag{41}$$

Where G_{ij} denotes the real part of the Y-parameter Y_{ij}, i.e. $Y_{ij} = G_{ij} + jB_{ij}$.
For M to be real, the Y-parameters of the device must satisfy the condition:

$$1 - \frac{Re(Y_{12}Y_{12})}{G_{11}G_{22}} - \left[\frac{Im(Y_{12}Y_{12})}{2G_{11}G_{22}}\right]^2 > 0 \tag{42}$$

Simultaneous conjugate matching (Y-parameters)

The optimum source and load admittance terminations, Y_{mS} and Y_{mL}, that result in G_{max}, can be shown to be:

$$Y_{mS} = G_{11}M + j\left[\frac{Im(Y_{21}Y_{12})}{2G_{22}} - B_{11}\right] \tag{43}$$

and

$$Y_{mL} = G_{22}M + j\left[\frac{Im(Y_{21}Y_{12})}{2G_{11}} - B_{22}\right] \tag{44}$$

A device that satisfies condition (42) and can therefore be simultaneously conjugately matched at both ports is referred to as being *unconditionally stable*.

Table of Contents

Stability in terms of immittance parameters

The Linvill stability factor is defined as :

$$C = \frac{|Y_{12}Y_{21}|}{2Re(Y_{11})Re(Y_{22}) - Re(Y_{12}Y_{21})} \tag{45}$$

The device is unconditionally stable for $0 < C < 1$. The inverse of the Linvill stability factor is called the *Rollett stability factor*, K, which tends to be more commonly used:

$$K = \frac{2Re(Y_{11})Re(Y_{22}) - Re(Y_{12}Y_{21})}{|Y_{12}Y_{21}|} \tag{46}$$

It is the Rollett stability factor that the reader is more likely to encounter in the literature. A necessary but not sufficient condition for unconditional stability requires that $K > 1$. If $K < 1$, the device is referred to as *potentially unstable* and may oscillate with some combinations of passive source and load immittances. This does not mean that the device cannot be used as an amplifier, it simply means that the source and load terminations must be carefully chosen to lie outside regions of instability.

Table of Contents

Stability in terms of S-parameters

We introduced the concept of active two-port stability in terms of immittance parameters in the preceding slides.

We now proceed to address the problem of stability from the S-parameter perspective by applying the following two categories of two-port stability :

- Unconditional Stability :A two-port is said to be inherently stable or unconditionally stable at a given frequency if no combination of passive terminations exists which will induce it to oscillate at that frequency.
- Conditional Stability (potential instability) : A two-port is said to be *potentially unstable* or *conditionally stable* at a given frequency if a combination of passive terminations can be found that will induce sustained oscillation at that frequency.

For potentially unstable devices there will be some combination of source and load terminations that will make the device unstable, meaning that there is a passive value of Γ_L that will result in $|\Gamma_{in}| > 1$ and vice versa.

We can show that the boundaries between stable and unstable regions on the source and load planes are in the form of a circle, known as a *stability circle*. The centres and radii of the stability circles are functions of the device S-parameters[6] .

Stability circles

Considering figure 9, and looking first at the load reflection coefficient plane, we define values of load termination which cause instability as those which cause the magnitude of the input reflection coefficient of the device to be greater than or equal to unity [7].

The input reflection coefficient of any two-port with the output port terminated with a load Γ_L is given by :

$$\Gamma_{in} = S_{11} + \frac{S_{12}S_{21}\Gamma_L}{1 - S_{22}\Gamma_L} = \frac{S_{11} - \Delta\Gamma_L}{1 - S_{22}\Gamma_L} \tag{47}$$

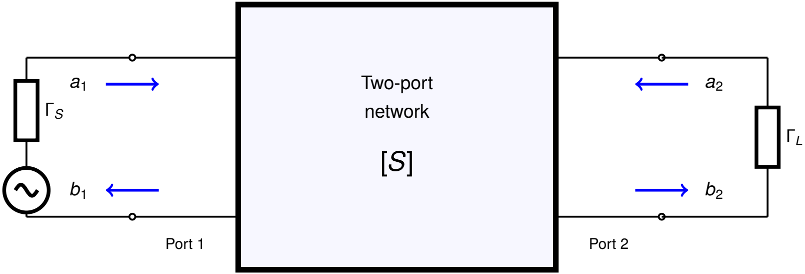


Figure 5 : Two-port network with arbitrary source and load

Stability circles : load plane

If we set $|\Gamma_{in}| = 1$ in (47) we establish a boundary in the Γ_L plane that separates stable from unstable regions. From (47) we can see that all values of Γ_L that lie on his boundary satisfy the following:

$$|S_{11} - \Delta\Gamma_L| = |1 - S_{22}\Gamma_L| \tag{48}$$

Squaring both sides of (48), collecting terms, and simplifying, results in :

$$|S_{11}| + |\Gamma_L|^2(|\Delta|^2 - |S_{22}|^2) + \Gamma_L C_2^* + \Gamma_L^* C_2 = 1 \tag{49}$$

Which can be written as :

$$|\Gamma_L|^2 - \Gamma_L \frac{C_2^*}{D_2^2} - \Gamma_L^* \frac{C_2}{D_2^2} = \frac{|S_{11}|^2 - 1}{D_2^2} \tag{50}$$

Where :

$$C_2 = S_{22} - \Delta S_{11}^* \tag{51}$$

$$D_2 = |S_{22}|^2 - |\Delta|^2 \tag{52}$$

Stability circles : load plane

Adding $|C_2|^2/D_2^2$ to both sides of (50) results in:

$$|\Gamma_L|^2 + \frac{|C_2|^2}{D_2^2} - \Gamma_L \frac{C_2^*}{D_2^2} - \Gamma_L^* \frac{C_2}{D_2^2} = \frac{(|S_{11}|^2 - 1)D_2^2 + |C_2|^2}{D_2^2} \tag{53}$$

Equation (53) is of the form of a circle in the Γ_L plane, i.e.:

$$|\Gamma_L - C_{SL}|^2 = |\gamma_{SL}|^2 \tag{54}$$

By comparing (53) and (54) we can derive the centre of the load plane stability circle as:

$$\boxed{C_{SL} = \frac{C_2^*}{|S_{22}|^2 - |\Delta|^2}} \tag{55}$$

Stability circles : load plane

By comparing (53) and (54) we can derive the radius of the load plane stability circle as:

$$\gamma_{SL} = \frac{\sqrt{(|S_{11}|^2 - 1)(|S_{22}|^2 - |\Delta|^2) + |C_2|^2}}{|S_{22}|^2 - |\Delta|^2} \tag{56}$$

Equation (56) may be simplified to yield:

$$\boxed{\gamma_{SL} = \frac{|S_{12}S_{21}|}{|S_{22}|^2 - |\Delta|^2}} \tag{57}$$

Stability circles : source plane

We can also define source plane stability circles by considering the boundary of load plane stability defined by $|\Gamma_{out}| = 1$ where Γ_{out} is defined by:

$$\Gamma_{out} = S_{22} + \frac{S_{12}S_{21}\Gamma_S}{1 - S_{11}\Gamma_S} = \frac{S_{22} - \Delta\Gamma_S}{1 - S_{11}\Gamma_S} \tag{58}$$

Following a similar analysis as above, starting with (58), we can derive the centre and radius of the source plane stability circle as follows:

$$\boxed{C_{SS} = \frac{C_1^*}{|S_{11}|^2 - |\Delta|^2}} \tag{59}$$

and

$$\boxed{\gamma_{SS} = \frac{|S_{12}S_{21}|}{|S_{11}|^2 - |\Delta|^2}} \tag{60}$$

Where :

$$C_1 = S_{11} - \Delta S_{22}^* \tag{61}$$

The stability circles represent the boundaries of permitted terminations in their respective reflection coefficient planes if stable operation is to be assured.

Stability circles : identifying the stable region

To determine whether the stable region is represented by the interior or exterior of the stability circle we consider what happens when we set $\Gamma_S = 0$, which, from (58), will result in $|\Gamma_{out}| = |S_{22}|$.

If $|S_{22}| < 1$ for that particular device, then the origin of the source plane Smith Chart must lie in the stable region. In other words, if $|S_{22}| < 1$ AND the source plane stability circle encompasses the origin, then the inside of the stability circle will represent the stable region.

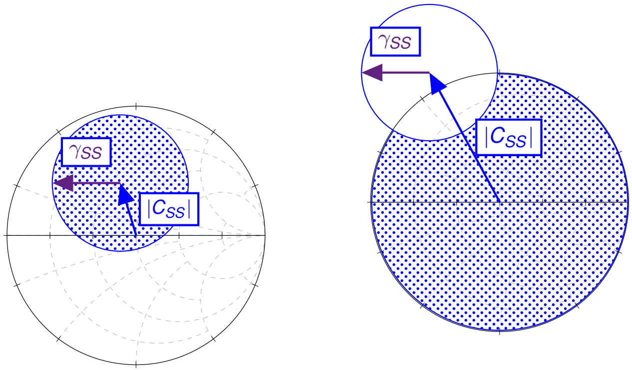
If $|S_{22}| < 1$ and the source plane stability circle does not encompass the origin, then the inside of the stability circle will represent the unstable region.

In general, therefore, the necessary condition for the source plane stability circle to encompass the origin is:

$$\gamma_{SS} > |C_{SS}| \tag{62}$$

If $|S_{22}| > 1$, however, the situation just described will be reversed. If $|S_{22}| > 1$ and the source plane stability circle does not encompass the origin then the stable region is represented by the inside of the stability circle.

Source plane stability circles ($|S_{22}| < 1$)

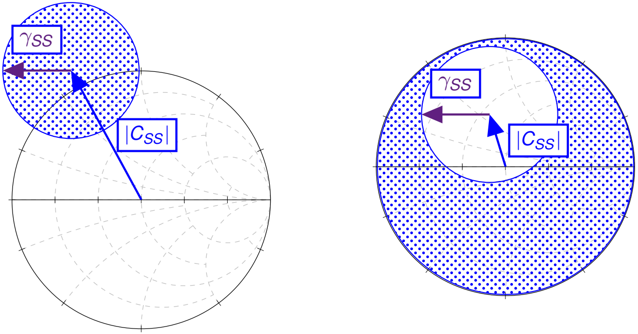


(a) $|S_{22}| < 1$, $\gamma_{SS} > |C_{SS}|$

(b) $|S_{22}| < 1$, $\gamma_{SS} < |C_{SS}|$

Figure 6 : Source plane stability circles (stable region shaded)

Source plane stability circles ($|S_{22}| > 1$)



(a) $|S_{22}| > 1, \gamma_{SS} < |C_{SS}|$

(b) $|S_{22}| > 1, \gamma_{SS} > |C_{SS}|$

Figure 7 : Source plane stability circles (stable region shaded)

Stability criteria in terms of S-parameters

The Rollett stability factor, K, was derived in terms of immittance parameters in previous slides. It can equally be defined in terms of S-parameters by considering the stability circles for an unconditionally stable device.

Figure 8 shows an example of a source plane stability circle lying entirely outside the unit circle in the Γ_S plane, meaning that there are no terminations in the unstable (shaded) region that can be realised with a passive source termination.

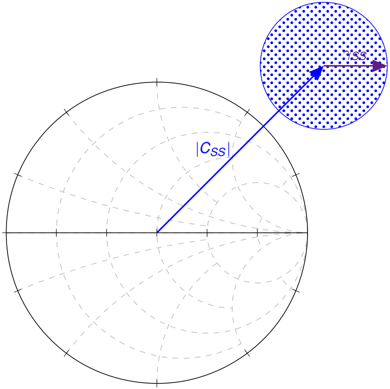


Figure 8 : Stability circle for an unconditional stable device

Stability criteria in terms of S-parameters

From figure 8 we can define the source plane stability criterion as :

$$\gamma_{SS} - |C_{SS}| > 1 \tag{63}$$

Substituting (55) and (56) into (63) results in :

$$\frac{|S_{22}^2 - \Delta S_{11}^*| - |S_{21}S_{12}|}{|S_{22}|^2 - |\Delta|^2} > 1 \tag{64}$$

Rearranging (64) and squaring both sides :

$$||S_{22}^2 - \Delta S_{11}^*| - |S_{12}S_{21}||^2 > ||S_{22}|^2 - |\Delta|^2|^2 \tag{65}$$

Expanding (65) results in :

$$2|S_{12}S_{21}||S_{22}^2 - \Delta S_{11}^*| < |S_{22}^2 - \Delta S_{11}^*|^2 + |S_{12}S_{21}|^2 - ||S_{22}|^2 - |\Delta|^2|^2 \tag{66}$$

Stability criteria in terms of S-parameters

We note that the term $|S_{11} - S_{22}\Delta|^2$ can be expressed as :

$$|S_{22}^2 - \Delta S_{11}^*|^2 = |S_{12}S_{21}|^2 + (1 - |S_{11}|^2)(|S_{22}|^2 - |\Delta|^2) \tag{67}$$

Substituting (67) into (66) yields :

$$(|S_{22}|^2 - |\Delta|^2)^2 \left[\left[(1 - |S_{11}|^2) - (|S_{22}|^2 - |\Delta|^2) \right] - 4|S_{12}S_{21}|^2 \right] > 0 \tag{68}$$

Which can be simplified to yield the following :

$$|S_{21}S_{12}| < 1 - |S_{11}|^2 - |S_{22}|^2 + |\Delta|^2 \tag{69}$$

This leads to the definition of the Rollett stability factor as :

$$\boxed{K = \frac{1 - |S_{11}|^2 - |S_{22}|^2 + |\Delta|^2}{2|S_{21}S_{12}|}} \tag{70}$$

and the Rollett stability criterion, which is :

$$K > 1 \tag{71}$$

A similar analysis for the load plane yields a result identical to equation (69).

Stability criteria in terms of S-parameters

The complete stability criterion in terms of S-parameters, as published by Rollett, takes into account the magnitude of S_{11} and S_{22}[8], and can be stated as the following simultaneous conditions:

$$\left.\begin{aligned} K &> 1 \\ |S_{11}| &< 1 \\ |S_{22}| &< 1 \end{aligned}\right\} \tag{72}$$

Woods[10] later showed that the Rollett criteria of (72) are insufficient to prove stability, and proposed a new set of stability criteria, namely :-

$$\left.\begin{aligned} K &> 1 \\ &\textit{and} \\ |\Delta| &< 1 \end{aligned}\right\} \tag{73}$$

Where : $\Delta = S_{11}S_{22} - S_{12}S_{21}$

Edwards and Sinsky Stability criteria

Due to the insufficiency of the Rollett stability criteria, new stability criterion has been growing in popularity. This is called the 'Edwards-Sinsky' criterion, after the authors who first published it in a 1992 paper[4].
Edwards and Sinsky employ a geometrical method to arrive at two stability parameters, referred to as μ_1 and μ_2, which are defined as follows :

$$\boxed{\mu_1 = \frac{1 - |S_{11}|^2}{|S_{22} - \Delta S_{11}^*| + |S_{12}S_{21}|}} \tag{74}$$

$$\boxed{\mu_2 = \frac{1 - |S_{22}|^2}{|S_{11} - \Delta S_{22}^*| + |S_{12}S_{21}|}} \tag{75}$$

μ_1 measures the radius from the center of the Smith Chart (Z_o) to the nearest unstable point in the output plane. μ_2 does the same for the input plane. In order for the unstable region to be outside the unit circle, the Edwards-Sinsky stability criterion are therefore :

$$\mu_1 > 1 \tag{76}$$

and simultaneously

$$\mu_2 > 1 \tag{77}$$

Edwards and Sinsky showed that these two criteria are equivalent, i.e. either $\mu_1 > 1$ or $\mu_2 > 1$ are sufficient to prove that an active two-port network is unconditionally stable.

Table of Contents

Power gain in terms of S-parameters

We will now revisit the discussion of power gain from the S-parameter perspective.

Consider the active two-port of figure 9, terminated with an arbitrary source and load, described by their reflection coefficients, Γ_S and Γ_L.

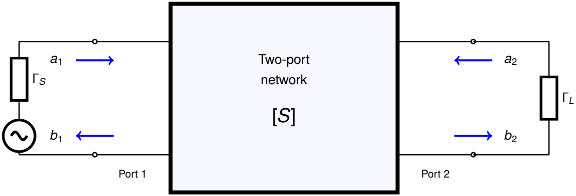


Figure 9 : Two-port network with arbitrary source and load

Power gain in terms of S-parameters

The various powers available and delivered at the input and output ports of a two-port device are defined, in terms of the power waves shown in figure 9, as follows :

- Power absorbed by the amplifier input (P_{in}) :
  This is the power actually delivered to the input port of the amplifier, irrespective of whether the source is conjugately matched to the input port. This is defined as:

$$P_{in} = \frac{1}{2}|a_1|^2 - \frac{1}{2}|b_1|^2 = \frac{1}{2}|a_1|^2(1 - |\Gamma_{in}|^2) \tag{78}$$

- Power available from the generator (P_{AVS}) :
  This is the maximum power delivered from the source when it is conjugately matched to the input port, i.e.:

$$P_{AVS} = \frac{1}{2}|b_S|^2 - \frac{1}{2}|a_S|^2 = \frac{1}{2} \cdot \frac{|b_S|^2}{(1 - |\Gamma_S|^2)} \tag{79}$$

- Power absorbed by the load (P_L) :
  This is the power actually delivered to the load, irrespective of whether the load is conjugately matched to the output of the two-port. This is defined as the difference between incident and reflected power at the load, i.e.:

$$P_L = \frac{1}{2}|b_2|^2 - \frac{1}{2}|a_2|^2 = \frac{1}{2}|b_2|^2(1 - |\Gamma_L|^2) \tag{80}$$

- Power available from the amplifier output (P_{AVN}) :
  This is the maximum power available from the amplifier output when it is conjugately matched to the load:

$$P_{AVN} = \frac{|v_2|^2}{4Re(Z_{out})} \tag{81}$$

Transducer Power gain in terms of S-parameters

The Transducer Power Gain, G_T, is the most general case which applies when source and load terminations are arbitrarily chosen (i.e. not necessarily matched to the respective ports). From (17) and employing the power definitions (79) and (80) we can write :

$$G_T = \frac{P_L}{P_{AVS}} = \frac{|b_2|^2}{|b_S|^2} \cdot (1 - |\Gamma_L|^2)(1 - |\Gamma_S|^2) \tag{82}$$

Figure 10 represents the signal flow within the two-port with load.

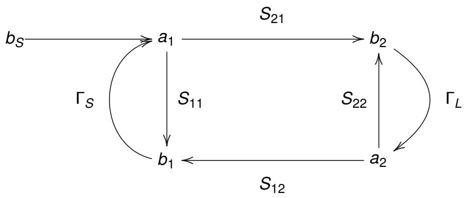


Figure 10 : Mason's rule applied to a two-port with load

Transducer Power gain in terms of S-parameters

Applying Mason's rule to figure 10 we get[5]:

$$\frac{|b_2|^2}{|b_S|^2} = \frac{S_{21}}{1 - (S_{11}\Gamma_S + S_{22}\Gamma_L + S_{21}\Gamma_L S_{12}\Gamma_S) + S_{11}\Gamma_S S_{22}\Gamma_L} \tag{83}$$

The denominator of (83) can be simplified to give:

$$\frac{|b_2|^2}{|b_S|^2} = \frac{S_{21}}{(1 - S_{11}\Gamma_S)(1 - S_{22}\Gamma_L) - S_{21}S_{12}\Gamma_S\Gamma_L} \tag{84}$$

Replacing $|b_2|^2/|b_S|^2$ in (82) by (84) gives :

$$\boxed{G_T = \frac{|S_{21}|^2(1 - |\Gamma_L|^2)(1 - |\Gamma_S|^2)}{|(1 - S_{11}\Gamma_S)(1 - S_{22}\Gamma_L) - S_{21}S_{12}\Gamma_S\Gamma_L|^2}} \tag{85}$$

Transducer Power gain in terms of S-parameters

Equation (85) expresses the transducer gain in terms of arbitrary source and load terminations. Alternatively we can express G_T in terms of Γ_{in} and Γ_{out} by further rearranging the denominator of (85) and employing (**??**) and (**??**). This results in two equivalent equations for transducer power gain, i.e. :

$$G_T = \frac{(1-|\Gamma_S|^2)}{|1-\Gamma_{in}\Gamma_S|^2}|S_{21}|^2\frac{(1-|\Gamma_L|^2)}{|1-S_{22}\Gamma_L|^2} \tag{86}$$

and

$$G_T = \frac{(1-|\Gamma_S|^2)}{|1-S_{11}\Gamma_S|^2}|S_{21}|^2\frac{(1-|\Gamma_L|^2)}{|1-\Gamma_{out}\Gamma_L|^2} \tag{87}$$

(86) and (87) are S-parameter equivalents of (30), having a similar structure consisting of the product of three factors, namely:

(i) an intrinsic gain component : $|S_{21}|^2$

(ii) a source mismatch factor : $\dfrac{(1-|\Gamma_S|^2)}{|1-S_{11}\Gamma_S|^2}$

(iii) a load mismatch factor : $\dfrac{(1-|\Gamma_L|^2)}{|1-S_{22}\Gamma_L|^2}$

Available Power gain in terms of S-parameters

The Available Power Gain, G_A, is a special case of Transducer gain with the output conjugately matched and the source termination as the variable. Setting $\Gamma_L = \Gamma_{out}^*$ in (87) gives:

$$G_A = \frac{|S_{21}|^2(1-|\Gamma_S|^2)}{|1-|\Gamma_{out}|^2||1-S_{11}\Gamma_S|^2} \tag{88}$$

We can rewrite (88) with Γ_S as the only dependent variable by replacing Γ_{out} in (88) by (**??**) as follows :

$$G_A = \frac{|S_{21}|^2(1-|\Gamma_S|^2)}{\left|1-\left|\frac{S_{22}-\Delta\Gamma_S}{1-S_{11}\Gamma_S}\right|^2\right||1-S_{11}\Gamma_S|^2} \tag{89}$$

Which simplifies to :

$$\boxed{G_A = \frac{|S_{21}|^2(1-|\Gamma_S|^2)}{|1-S_{11}\Gamma_S|^2-|S_{22}-\Delta\Gamma_S|^2}} \tag{90}$$

Operating Power gain in terms of S-parameters

The operating power gain is a special case of transducer gain with the input conjugately matched and the load termination as the variable. Setting $\Gamma_S = \Gamma_{in}^2$ in (86) gives:

$$G_o = \frac{|S_{21}|^2(1-|\Gamma_L|^2)}{||1-|\Gamma_{in}|^2||1-S_{22}\Gamma_L|^2} \tag{91}$$

We can rewrite (91) with Γ_L as the only dependent variable by replacing Γ_{in} in (91) by (**??**) as follows :

$$G_o = \frac{|S_{21}|^2(1-|\Gamma_L|^2)}{\left|1-\left|\frac{S_{11}-\Delta\Gamma_L}{1-S_{22}\Gamma_L}\right|^2\right||1-S_{22}\Gamma_L|^2} \tag{92}$$

Which simplifies to :

$$\boxed{G_o = \frac{|S_{21}|^2(1-|\Gamma_L|^2)}{|1-S_{22}\Gamma_L|^2-|S_{11}-\Delta\Gamma_L|^2}} \tag{93}$$

Summary of Power gains in terms of S-parameters

In the special case where we terminate the two-port in the system characteristic impedance, Z_o, used to measure the original S-parameter, we have $\Gamma_S = \Gamma_L = 0$, and the gains defined in (85), (90) and (93) are reduced to[2]:

$$\textit{Transducer Gain} : G_T = |S_{21}|^2 \tag{94}$$

$$\textit{Available Gain} : G_A = \frac{|S_{21}|^2}{1 - |S_{22}|^2} \tag{95}$$

$$\textit{Operating Gain} : G_o = \frac{|S_{21}|^2}{1 - |S_{11}|^2} \tag{96}$$

Maximum available gain and conjugate terminations

From (85) we can prove that G_T (and, by implication, G_A and G_o) is maximised when :

$$\Gamma_{in} = \Gamma_S^* \tag{97}$$

and, simultaneously:

$$\Gamma_{out} = \Gamma_L^* \tag{98}$$

By applying (97) and (98) to (**??**) and (**??**) we can define the optimum terminations by the following simultaneous equations:

$$\Gamma_S^* = S_{11} + \frac{S_{12}S_{21}\Gamma_L}{1 - S_{22}\Gamma_L} \tag{99}$$

$$\Gamma_L^* = S_{22} + \frac{S_{12}S_{21}\Gamma_S}{1 - S_{11}\Gamma_S} \tag{100}$$

Maximum available gain and conjugate terminations

By solving (99) and (100) simultaneously we obtain the expressions the values of source and load termination for simultaneous conjugate matching. These values are referred to as Γ_{ms} and Γ_{ml} and are defined as[1, 5]:

$$\boxed{\Gamma_{ms} = C_1^* \left[\frac{B_1 \pm \sqrt{B_1^2 - 4|C_1|^2}}{2|C_1|^2} \right]} \tag{101}$$

$$\boxed{\Gamma_{ml} = C_2^* \left[\frac{B_2 \pm \sqrt{B_2^2 - 4|C_2|^2}}{2|C_2|^2} \right]} \tag{102}$$

Where the variables B_1, B_2, C_1 and C_2 are defined by (**??**), (**??**), (61) and (51) respectively.

The negative sign in (101) applies if $B_1 > 0$ and the negative sign in (102) if $B_2 > 0$.

Maximum available gain and conjugate terminations

By substituting equations (101) and (102) into any one of the equations and by making use of equation (70) the value of the Maximum Available Gain(MAG) can be determined as[1, 5]:

$$\boxed{MAG = \frac{|S_{21}|}{|S_{12}|}\left[K \pm \sqrt{K^2 - 1}\right]} \quad (103)$$

Here the negative sign in (103) applies if $B_1 > 0$ and $B_2 > 0$. As an interesting aside, under these conditions the reverse isolation of the device is also at a maximum and is given by:

$$RI = \frac{|S_{12}|}{|S_{21}|}\left[K \pm \sqrt{K^2 - 1}\right] \quad (104)$$

With the same conditions applying to the polarity of the square root term in (104) as in (103). The maximum available gain is only realisable if the active device is unconditionally stable as defined in section **??**. If this is not the case, then we have to settle for the *Maximum Stable Gain* which is defined by [3]

$$\boxed{MSG = \frac{|S_{21}|}{|S_{12}|}} \quad (105)$$

Constant operating power gain circles (Load plane)

The operating power gain, G_o, is a function of the load termination, under the assumption that the input port match will always be conjugately matched (i.e. $\Gamma_S = \Gamma_{in}^*$). Recalling equation (93) for the operating power gain:

$$G_o = \frac{|S_{21}|^2(1-|\Gamma_L|^2)}{|1-S_{22}\Gamma_L|^2 - |S_{11}-\Delta\Gamma_L|^2}$$

We can rewrite this in the following form, by expanding the denominator :

$$G_o = \frac{|S_{21}|^2(1-|\Gamma_L|^2)}{1-|S_{11}|^2+|\Gamma_L|^2 D_2 - 2Re(C_2\Gamma_L)} \tag{106}$$

Where C_2 and D_2 are defined by:

$$\begin{aligned} C_2 =& S_{22} - \Delta S_{11}^* \\ D_2 =& (|S_{22}|^2 - |\Delta|^2) \end{aligned}$$

We will now define a normalised operating power gain parameter, g_o as :

$$g_o = \frac{G_o}{|S_{21}|^2} \tag{107}$$

Then, using (106) we can write :

$$g_o = \frac{1-|\Gamma_L|^2}{1-|S_{11}|^2+|\Gamma_L|^2 D_2 - 2Re(C_2\Gamma_L)} \tag{108}$$

Constant operating power gain circles (Load plane)

At this point, we recall that the equation of a circle on the Γ_L plane is of the form :

$$|\Gamma_L - C_{gL}|^2 = |\gamma_{gL}|^2 \tag{109}$$

Where C_{gL} is the centre and γ_{gL} is the radius of the constant operating gain circle. We can thus rearrange equation (108) to be in the form of (109), as follows :

$$\left|\Gamma_L - \frac{g_o C_2^*}{1 + g_o D_2}\right|^2 = \left|\frac{\sqrt{(1 - 2K|S_{12}S_{21}|g_o + |S_{12}S_{21}|^2 g_o^2)}}{1 + g_o D_2}\right|^2 \tag{110}$$

By comparing (110) with (109) we can discern the centres and radii of the constant operating power gain circles on the load reflection coefficient plane as follows:

$$\boxed{C_{gL} = \frac{g_o C_2^*}{1 + g_o D_2}} \tag{111}$$

$$\boxed{\gamma_{gL} = \frac{\sqrt{(1 - 2K|S_{12}S_{21}|g_o + |S_{12}S_{21}|^2 g_o^2)}}{1 + g_o D_2}} \tag{112}$$

Constant available power gain circles (Source plane)

We may need to determine the effect of varying the source reflection coefficient with the output port conjugately matched (i.e. $\Gamma_L = \Gamma_{out}^*$). In which case we need to use the available power gain, G_A.

$$G_A = \frac{|S_{21}|^2(1 - |\Gamma_S|^2)}{|1 - S_{11}\Gamma_S|^2 - |S_{22} - \Delta\Gamma_S|^2}$$

We can expand the denominator and rewrite (90) in the form :

$$G_A = \frac{|S_{21}|^2(1 - |\Gamma_S|^2)}{1 - |S_{22}|^2 + |\Gamma_S|^2 D_1 - 2Re(C_1\Gamma_S)} \qquad (113)$$

Where C_1 and D_1 are defined by :

$$\begin{aligned} C_1 &= S_{11} - \Delta S_{22}^* \\ D_1 &= (|S_{11}|^2 - |\Delta|^2) \end{aligned}$$

We will define the normalised available power gain parameter, g_a as :

$$g_a = \frac{G_A}{|S_{21}|^2} \qquad (114)$$

Then, using (113) we can write :

$$g_a = \frac{1 - |\Gamma_S|^2}{1 - |S_{22}|^2 + |\Gamma_S|^2 D_1 - 2Re(C_1\Gamma_S)} \qquad (115)$$

Constant available power gain circles (Source plane)

Which we can rearrange into the form of the equation of a circle on the Γ_S, similar to (109) i.e.:

$$|\Gamma_S - C_{gS}|^2 = |\gamma_{gS}|^2 \qquad (116)$$

Rearranging (115) to be in the form of (116) we get :

$$\left|\Gamma_S - \frac{g_p C_1^*}{1+g_a D_1}\right|^2 = \left|\frac{\sqrt{(1-2K|S_{12}S_{21}|g_a + |S_{12}S_{21}|^2 g_a^2)}}{1+g_a D_1}\right|^2 \qquad (117)$$

Once again, by comparing (117) with (116) we can determine the centres and radii of the constant available power gain circles on the source reflection coefficient plane as follows:

$$\boxed{C_{gS} = \frac{g_a C_1^*}{1+g_a D_1}} \qquad (118)$$

$$\boxed{\gamma_{gS} = \frac{\sqrt{(1-2K|S_{12}S_{21}|g_a + |S_{12}S_{21}|^2 g_a^2}}{1+g_a D_1}} \qquad (119)$$

General observations on Constant power gain circles

We make the following general observations on the properties of the gain circle equations (111), (112), (118) and (119), as follows:

1. We note that the centres of load plane constant gain circles always lie on a line drawn between the point C_2^* and the origin of the load plane. Similarly, the centres of source plane constant gain circles always lie on a line drawn between the point C_1^* and the origin of the source plane.
2. The radius of constant gain circles decreases with increasing gain. In case of an unconditionally stable device, the maximum gain is therefore achieved when $\gamma_{gL} = 0$ or $\gamma_{gS} = 0$. Applying these conditions to either (112) or (119) results in :

$$g_{max} = \frac{1}{|S_{12}S_{21}|}(K - \sqrt{K^2 - 1}) \tag{120}$$

   Which, applying (114), gives :

$$G_{max} = \frac{|S_{21}|}{|S_{12}|}(K - \sqrt{K^2 - 1}) \tag{121}$$

   Readers will note that equation (120) corresponds to maximum available gain (MAG) derived for an unconditional stable amplifier in (103).
3. In case of a potentially unstable device, i.e. when $G_o = \infty$, and $G_A = \infty$, the constant gain circles described by equations (111), (112), (118) and (119) become equal to the source and load plane stability circles, described by (55), (57), (59) and (60).

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